

Holography, Unfolding and Higher-Spin Theory

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Abstract

Holographic duality is argued to relate classes of models that have equivalent unfolded formulation, hence exhibiting different space-time visualizations for the same theory. This general phenomenon is illustrated by the AdS_4 higher-spin gauge theory shown to be dual to the theory of $3d$ conformal currents of all spins interacting with $3d$ conformal higher-spin fields of Chern-Simons type. Generally, the resulting $3d$ boundary conformal theory is nonlinear, providing an interacting version of the $3d$ boundary sigma model conjectured by Klebanov and Polyakov to be dual to the AdS_4 HS theory in the large N limit. Being a gauge theory it escapes the conditions of the theorem of Maldacena and Zhiboedov, which force a $3d$ boundary conformal theory to be free. Two reductions of particular higher-spin gauge theories where boundary higher-spin gauge fields decouple from the currents and which have free boundary duals are identified. Higher-spin holographic duality is also discussed for the cases of AdS_3/CFT_2 and duality between higher-spin theories and nonrelativistic quantum mechanics. In the latter case it is shown in particular that (dS) AdS geometry in the higher-spin setup is dual to the (inverted) harmonic potential in the quantum-mechanical setup.

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1 Introduction

Higher-spin (HS) gauge theories describe interactions of massless fields of all spins. First example of fully nonlinear HS theory was given in the $4d$ case in [1], while its modern formulation was worked out in [2] (see [3] for a review). A specific property of HS gauge theories is that consistent interactions of propagating massless fields require a curved background which provides a length scale in HS interactions that contain higher derivatives. $(A)dS$ is the most symmetric curved background compatible with HS interactions. The AdS_4 HS model is the simplest nontrivial in the sense that $d = 4$ is the lowest dimension where HS massless fields propagate. After the AdS/CFT correspondence conjecture was put forward in [4, 5, 6], the fact that HS theories are most naturally formulated in the AdS background was conjectured to play a role in the context of AdS/CFT correspondence [7, 8, 9, 10, 11]. This expectation conforms the fundamental result of Flato and Fronsdal [12] on the relation between tensor products of $3d$ conformal fields (singletons) and infinite towers of $4d$ massless fields that appear in HS theories.

In the important work of Klebanov and Polyakov [13] it was argued that the HS gauge theory of [2] should be dual to the $3d$ $O(N)$ sigma model in the $N \rightarrow \infty$ limit. The Klebanov-Polyakov conjecture was checked by Giombi and Yin in [14, 15] where it was shown in particular how the bulk computation in HS gauge theory reproduces at least some of conformal correlators in the free $3d$ theory. (For related computations in free HS theory see also [16, 17, 18, 19].) Recently, Maldacena and Zhiboedov [20] addressed the question on restrictions imposed on a boundary $3d$ conformal theory by HS conformal symmetries. Assuming very general conditions on conformal theory which included unitarity, locality and conformal operator product algebra they were able to show that a conformal HS theory, that possesses a HS conserved current, should be free. This conclusion seemingly suggests that any AdS_4 HS theory should be equivalent to a free boundary theory at least in the most symmetric vacuum.

The primary motivation for this work was to analyze directly a $3d$ dual of the AdS_4 HS theory by means of the unfolded dynamics approach which makes this analysis straightforward, describing evolution with respect to different coordinates as independent mutually commuting flows. This property allows us to obtain $3d$ field equations from $4d$ HS equations simply by reducing four space-time coordinates of AdS_4 to three, relating directly two seemingly different theories in a way anticipated from AdS/CFT correspondence.

An important ingredient of the duality is that unfolded equations for $3d$ conserved currents result from the $3d$ reduction of $4d$ unfolded massless equations. The key observation is based on the interpretation of currents as rank-two fields within the approach of [21] where it was shown that conserved currents, built from the free fields described by $C(Y|X)$ where Y are auxiliary spinor (twistor) variables while X describe space-time coordinates, are described by the fields $J(Y^i|X)$ with the doubled number of spinor variables Y^i , $i = 1, 2$. In particular, free $3d$ massless fields are described by functions of two-component spinors y^α and space-time coordinates $x^{\alpha\beta} = x^{\beta\alpha}$, where $\alpha, \beta = 1, 2$, while $3d$ conserved currents $J(y^i|x)$ depend on a pair of spinors y_α^i . On the other hand, a $4d$ massless field is described

by a field $C(Y|X)$ where $Y = (y^\alpha, \bar{y}^{\dot{\alpha}})$ are complex two-component spinors and $X = x^{\alpha\dot{\alpha}}$ encode four real coordinates in the form of 2×2 Hermitian matrices. It suffices to modify hermiticity conditions for two-component spinors to identify $4d$ massless fields with $3d$ conformal conserved currents. The pullback of the known nonlinear $4d$ massless field equations to a $3d$ subspace $\Sigma^3 \in AdS_4$ gives nonlinear $3d$ equations which describe interactions of conformal current fields with $3d$ conformal HS gauge fields. Precise identification only requires an appropriate change of the reality conditions and transition to the conformal frame where conformal symmetries are manifest.

In this setup holography takes place for generic $3d$ surface Σ . However, the map from the AdS frame to the conformal frame, where conformal symmetries are properly realized, turns out to be nonlocal for general Σ . In the unfolded formulation of HS theory, this map has simple meaning in terms of noncommutative twistor variables Y , describing transition from the Weyl star product in the bulk theory to the normal ordered one in the conformal frame. However, in space-time terms it may look obscure for general Σ . Remarkably, in the limit where Σ is AdS_4 infinity, the correspondence between the two frames becomes local and very simple, directly identifying $4d$ massless fields with (sources for) $3d$ currents in accordance with the original AdS/CFT prescription of [5, 6].

The conclusion that nonlinear AdS_4 HS gauge theory is dual to a nonlinear $3d$ theory does not contradict to Maldacena-Zhiboedov theorem [20] because the boundary theory turns out to be a gauge theory with currents interacting through $3d$ Chern-Simons conformal HS gauge fields. As such it escapes at least one of the assumptions of unitarity, locality and/or conformal invariance. Indeed, gauge degrees of freedom correspond to null states, while a gauge fixing procedure breaks covariance and/or locality. There are, however, two special configurations for AdS_4 HS fields that correspond to Dirichlet and Neumann boundary conditions in the so-called A and B models, where the $3d$ superconformal HS gauge fields decouple from currents. These correspond to the free bosonic and fermionic boundary theories in accordance with Klebanov-Polyakov [13] and Sezgin-Sundell [22] conjectures as well as Maldacena-Zhiboedov theorem [20].

Analysis of HS holography within unfolded dynamics shows that phenomenon of holographic duality is absolutely general, taking place in any AdS theory. To make the correspondence manifest, the theory in question has to be reformulated in the unfolded form. This makes the correspondence to large extent tautological via reduction of space-time coordinates in the unfolded theory. More precisely, the unfolding procedure effectively reformulates a theory in terms of appropriate (generalized) twistor variables Y rather than directly in terms of space-time coordinates X . In this setup, holography relates theories in different space-times \mathbf{M} (coordinates X), that have the same description in the twistor space \mathbf{T} (coordinates Y). Reformulation of a theory in the unfolded form makes it straightforward to identify its holographic duals by choosing different space-times \mathbf{M} for the same twistor model.

In the general setup of this paper, the large N parameter does not play any significant role. In HS theories it can be related to the HS coupling constant so that in the large N limit it brings the boundary theory to the free field limit. However, interactions with boundary HS gauge fields should be taken into account in the analysis of subleading $1/N$ corrections.

Although in this paper we mainly discuss the correspondence at the level of field equations because the appropriate action principle for HS gauge theories remains unknown (for some interesting conjectures see however [23, 24]) it should be stressed that general nature of holographic duality extends to the action level. Here we follow the analogy with the properties of conserved currents discussed in [25] where conserved charges in HS theory were represented in the form

$$q = \int_{\sigma} \Omega(Y|X), \quad (1.1)$$

where $\Omega(Y|X)$ is a closed p -form in some “correspondence” space \mathbf{C} with local coordinates Y (twistor space \mathbf{T}) and X (space-time \mathbf{M}). The charge q is independent of local variations of σ in \mathbf{C} while $\dim \mathbf{C}$ may be much larger than p . For example, in the case of [25], q could be represented either in the standard form of a space integral or as an integral in \mathbf{T} .

Since, as shown in [26], in the unfolded dynamics approach the concepts of conserved charge and action are similar, analogous representation is anticipated to be reachable for HS actions. If so, the generating functional in the bulk theory

$$Z_{bulk}(\phi_{boun}) = \int \exp iS(\phi(Y|X)) \quad (1.2)$$

with boundary conditions ϕ_{boun} for the bulk dynamical variables $\phi(Y|X)$ can be represented in many equivalent ways. If the integration is over space-time M_{bulk} this leads to the *AdS/CFT* prescription. If it is over some cycle in the twistor space \mathbf{T} or in $M_{bound} \times \mathbf{T}$ this provides an independent definition of the boundary theory. (Note that M_{bound} itself has too small dimension to support the integral).

An interesting output of the analysis of this paper is that, at least for the HS models in question, it is most convenient to formulate them in the doubled *AdS* space where the original *AdS* boundary $z = 0$ is identified with the invariant surface for the reflection automorphism P that maps one copy of *AdS* to another via reflection $P(z) = -z$. As a result, no boundary conditions at $z = 0$ should be imposed to define the action as the integral over the doubled *AdS* space-time. In this setup, holographic duality relates a bulk theory in the doubled *AdS* space with the “boundary theory” where all possible types of boundary fields $\phi_{bound}(\mathbf{x})$ contribute. Values of $\phi_{bound}(\mathbf{x})$ at $z = 0$ determine all fields in the (doubled) bulk and hence values of the respective action functionals $S(\phi_{bound})$. We believe that this doubling trick should have a wide area of applicability in HS theories and beyond.

The paper is organized as follows. In Section 2, we recall relevant elements of the unfolded dynamics approach. Interpretation of holography in terms of unfolded dynamics is discussed in Section 3. Unfolded equations for massless fields in *AdS*₄ are recalled in Section 4 while unfolded formulation for 3d conserved currents is summarized in Section 5. In Section 6, we recall nonlinear HS equations in *AdS*₄ also discussing in Section 6.2 their extension with spinor coordinates. In Section 7, holographic duality between HS theory in *AdS*₄ and 3d conformal HS theory is discussed in general terms. Its detailed analysis in terms of familiar Poincaré coordinates is presented in Section 8. General structure of nonlinear 3d conformal HS theory is discussed in Section 9. Boundary conditions, the construction of the doubled

AdS bulk space and reductions of particular HS theories, associated with free boundary theories, are considered in Section 10. *AdS*₃/*CFT*₂ HS correspondence is briefly discussed in Section 11. Duality between HS gauge theories and nonrelativistic quantum mechanics is considered in Section 12. Aspects of the off-shell extension of on-shell unfolded theories are considered in Section 13. Conclusions and perspectives are presented in Section 14.

2 Unfolded dynamics

2.1 Unfolded equations

Unfolded formulation is a multidimensional coordinate independent generalization of the first-order formulation

$$dt \frac{\partial}{\partial t} q^\alpha = G^\alpha(q), \quad G^\alpha(q) = e F^\alpha(q) \quad (2.1)$$

available for any system of ordinary differential equations by adding auxiliary variables associated with higher derivatives of the dynamical variables of the original system. Here e is a einbein 1-form that can be identified with dt because $1d$ geometry is flat.

Let M^d be a d -dimensional space-time manifold with coordinates $x^{\underline{n}}$ ($\underline{n} = 0, 1, \dots, d-1$). By unfolded formulation of a linear or nonlinear system of partial differential equations (PDE) in M^d we mean its reformulation in the first-order form [27]

$$dW^\Omega(x) = G^\Omega(W(x)), \quad (2.2)$$

where $d = dx^{\underline{n}} \frac{\partial}{\partial x^{\underline{n}}}$ is the exterior differential in M^d , $W^\Omega(x)$ is a set of degree p_Ω differential forms and $G^\Omega(W)$ is some degree $p_\Omega + 1$ function of W^Λ

$$G^\Omega(W) = \sum_{n=1}^{\infty} f^\Omega_{\Lambda_1 \dots \Lambda_n} W^{\Lambda_1} \wedge \dots \wedge W^{\Lambda_n}, \quad (2.3)$$

where the coefficients $f^\Omega_{\Lambda_1 \dots \Lambda_n}$ satisfy the (anti)symmetry condition

$$f^\Omega_{\Lambda_1 \dots \Lambda_k \Lambda_l \dots \Lambda_n} = (-1)^{p_{\Lambda_k} p_{\Lambda_l}} f^\Omega_{\Lambda_1 \dots \Lambda_l \Lambda_k \dots \Lambda_n} \quad (2.4)$$

(extension to the case with additional boson-fermion grading is straightforward) and $G^\Omega(W)$ satisfies the condition

$$G^\Lambda(W) \wedge \frac{\partial G^\Omega(W)}{\partial W^\Lambda} = 0 \quad (2.5)$$

equivalent to the following generalized Jacobi identity on the structure coefficients

$$\sum_{n=0}^m (n+1) f^\Phi_{[\Lambda_1 \dots \Lambda_{m-n}} f^\Omega_{\Phi \Lambda_{m-n+1} \dots \Lambda_m\}} = 0, \quad (2.6)$$

where the brackets $[\]$ denote an appropriate (anti)symmetrization of all indices Λ_i . Strictly speaking, generalized Jacobi identities (2.6) have to be satisfied at $p_\Omega < d$ since any $(d+1)$ -form in M^d is zero. Given solution of (2.6) it defines a free differential algebra [28, 29, 30, 31]. We call a free differential algebra *universal* [32, 26] if the generalized Jacobi identity holds independently of a particular value of space-time dimension. All HS free differential algebras relevant to HS theories including those discussed in this paper are universal. For example, the $1d$ system (2.1) is universal. Here the condition (2.5) trivializes because $e \wedge e = 0$, *i.e.*, any function $F^\alpha(q)$ is allowed. The generalized Jacobi identity is obeyed for any number of coordinates of the ambient space (*i.e.*, $dx^{\underline{m}}$), since $e = dx^{\underline{m}}e_{\underline{m}}$ carries no fiber indices.

Condition (2.5), which can equivalently be rewritten as

$$Q^2 = 0, \quad Q = G^\Omega(W) \frac{\partial}{\partial W^\Omega}, \quad (2.7)$$

guarantees formal consistency of the unfolded system (2.2) which can now be put into the form

$$dF(W(x)) = Q(F(W(x))) \quad (2.8)$$

with $d^2 = 0$ for all $F(W)$. Unfolded equations in the form (2.8) are analogous to $1d$ Hamiltonian equations.

Equation (2.2) is invariant under the gauge transformation

$$\delta W^\Omega = d\varepsilon^\Omega + \varepsilon^\Lambda \frac{\partial G^\Omega(W)}{\partial W^\Lambda}, \quad (2.9)$$

where the gauge parameter $\varepsilon^\Omega(x)$ is a $(p_\Omega - 1)$ -form. (0-forms among W^Ω do not have associated gauge parameters.)

2.2 Examples

2.2.1 Vacuum

Important example of unfolded equations is provided by Maurer-Cartan equations. Let h be a Lie algebra with some basis $\{T_\alpha\}$. Let $w = w^\alpha T_\alpha$ be a h -valued 1-form. Setting $G(w) = -w \wedge w \equiv -\frac{1}{2}w^\alpha \wedge w^\beta [T_\alpha, T_\beta]$, Eq. (2.2) with $W = w$ becomes the flatness condition

$$dw + w \wedge w = 0. \quad (2.10)$$

Eq. (2.5) amounts to the Jacobi identity for h . Eq. (2.9) gives the usual gauge transformation

$$\delta w = D_0 \varepsilon := d + [w, \varepsilon]. \quad (2.11)$$

Usually, the zero-curvature equations (2.10) describe background geometry in a coordinate independent way. For example, let h be Poincaré algebra with the gauge fields

$$w(x) = e^n(x)P_n + \omega^{nm}(x)L_{nm}, \quad (2.12)$$

where P_n and L_{nm} are generators of translations and Lorentz transformations. The gauge fields $e^n(x)$ and $\omega^{nm}(x)$ are identified with the frame 1-form and Lorentz connection, respectively (fiber Lorentz vector indices $m, n \dots$ and base indices $\underline{m}, \underline{n} \dots$ run from 0 to $d-1$ and are raised and lowered by the flat Minkowski metric). It is well known that the zero-curvature condition (2.10) for the Poincaré algebra amounts to the zero-torsion condition

$$R^n = de^n + \omega^n_m \wedge e^m = 0, \quad (2.13)$$

which expresses $\omega^{nm}(x)$ in terms of derivatives of the frame field, and the condition that Riemann tensor is zero

$$R^{mn} = d\omega^{mn} + \omega^m_k \wedge \omega^{kn} = 0, \quad (2.14)$$

which implies flat Minkowski geometry. As a result, at the condition that the matrix $e_{\underline{n}}^n(x)$ is nondegenerate, the zero-curvature condition (2.10) for the Poincaré algebra gives coordinate independent description of Minkowski space-time. Choosing a different Lie algebra h one can describe different background like, e.g., (anti-) de Sitter.

2.2.2 Linear fluctuations

If the set W^Ω contains some p -forms denoted by \mathcal{C}^i (e.g. 0-forms) and $G^i(W)$ is linear both in w and in \mathcal{C}^i

$$G^i = -w^\alpha (T_\alpha)^i_j \wedge \mathcal{C}^j, \quad (2.15)$$

relation (2.5) implies that $(T_\alpha)^i_j$ form some representation T of h , acting in a space V where \mathcal{C}^i is valued. The corresponding equation (2.2) is a covariant constancy condition

$$D_w \mathcal{C}^i = 0 \quad (2.16)$$

with $D_w \equiv d + w$ being the covariant derivative in the h -module V . For $G^i(w, \mathcal{C}^i)$ multilinear in the background connections w but still linear in dynamical fields \mathcal{C}^j , unfolded equations can be interpreted in terms of Chevalley-Eilenberg cohomology of h with coefficients in the infinite-dimensional modules carried by differential forms of different degrees among \mathcal{C}^j [33].

As an illustration, consider unfolded formulation of a scalar field. Following [34] we introduce the infinite set of 0-forms $C_{m_1 \dots m_n}(x)$ ($n = 0, 1, 2, \dots$), which are totally symmetric tensors $C_{m_1 \dots m_n} = C_{(m_1 \dots m_n)}$. The off-shell unfolded equations are

$$dC_{m_1 \dots m_n} = e^k C_{m_1 \dots m_n k} \quad (n = 0, 1, \dots), \quad (2.17)$$

where we use Cartesian coordinates with $D^L = d$. This system is formally consistent because application of d to the both sides of (2.17) gives zero by $e^k \wedge e^l = -e^l \wedge e^k$. Hence, the space V of 0-forms $C_{m_1 \dots m_n}$ forms some (infinite-dimensional) $iso(d-1, 1)$ -module. (Strictly speaking, one has to check that the equation is consistent for any flat $iso(d-1, 1)$ connection. It is not hard to see that this is indeed true.)

Let the scalar field $C(x)$ be identified with $C_{m_1 \dots m_n}(x)$ at $n = 0$. The first two equations of the system (2.17) read

$$\partial_{\underline{n}} C = C_{\underline{n}}, \quad \partial_{\underline{n}} C_{\underline{m}} = C_{\underline{mn}}, \quad (2.18)$$

where we have identified the world and tangent indices via $e_{\underline{m}}^m = \delta_{\underline{m}}^m$. The first of these equations tells us that $C_{\underline{n}}$ is the first derivatives of C . The second one identifies $C_{\underline{nm}}$ with the second derivative of C . All other equations in (2.17) express highest tensors in terms of the higher-order derivatives

$$C_{\underline{m}_1 \dots \underline{m}_n} = \partial_{\underline{m}_1} \dots \partial_{\underline{m}_n} C. \quad (2.19)$$

From Eq.(2.19) it is clear that the 0-forms $C_{\underline{n}_1 \dots \underline{n}_n}$ describe all derivatives of the dynamical field $C(x)$ including $C(x)$ itself. The system (2.17) is off-shell: it is equivalent to an infinite set of constraints, imposing no field equations on the dynamical field C .

To put the system on shell we impose an additional condition that all $C_{m_1 \dots m_n}(x)$ are traceless

$$C^k_{km_3 \dots m_n}(x) = 0. \quad (2.20)$$

This condition preserves consistency and puts the system on shell by virtue of Eq. (2.18).

2.3 Global symmetries

Maximally symmetric vacua of dynamical systems are described by vacuum connections that satisfy the flatness condition (2.10) for some Lie algebra h . Choosing some vacuum connection $w_0(x)$, global symmetry transformations that leave w_0 invariant are described by the parameters $\varepsilon_{gl}(x)$ that satisfy

$$D_0 \varepsilon_{gl} = 0. \quad (2.21)$$

Clearly, in the topologically trivial situation this equation has $\dim h$ independent solutions.

This simple observation immediately uncovers maximal symmetries of the linear unfolded equations of the form (2.16). Indeed, an h -module V can be treated as the $l^{max}(V)$ -module where $l^{max}(V)$ is the Lie algebra of commutators of $End V$. Indeed, since $h \in l^{max}(V)$, any flat connection w_0 of h can be interpreted as a flat connection of $l^{max}(V)$. Hence, $l^{max}(V)$ is the maximal symmetry of the linear unfolded equations with dynamical fields valued in V . As a result, via identification of V , unfolding of a dynamical system makes all its symmetries manifest.

Let W_0^Ω be some solution of the unfolded system (2.2), may be containing some nonzero p_Ω -forms with $p_\Omega \neq 1$. According to (2.9), symmetries of this solution are described by the symmetry parameters $\varepsilon_{gl}^\Omega(x)$ that satisfy

$$d\varepsilon_{gl}^\Omega + \varepsilon_{gl}^\Lambda \frac{\partial G^\Omega(W)}{\partial W^\Lambda} \Big|_{W=W_0} = 0. \quad (2.22)$$

Since equations (2.22) that contain $d\varepsilon_{gl}^\Omega$ are consistent as a consequence of the original unfolded equations they can be solved locally in terms of $\varepsilon_{gl}^\Omega(x_0)$ at any space-time point x_0 . Naively, it looks like this gives as many global symmetries as parameters ε_{gl}^Ω . However, this is not the case because the 0-form part of Eq. (2.22) may impose constraints on $\varepsilon_{gl}^\Omega(x)$

$${}^0\varepsilon_{gl}^\Lambda \frac{\partial G^\Omega(W)}{\partial {}^1W^\Lambda} \Big|_{W={}^0W_0} = 0, \quad (2.23)$$

where ${}^pW^\Omega$ denotes p -forms among W^Ω . This implies that nontrivial global symmetries should leave invariant vacuum values of 0-forms in the system. This restriction is very strong since, in the unfolded dynamics approach, nontrivial curvatures like Weyl tensor and its HS analogues are described by 0-forms. Most symmetric vacua are associated with those solutions where all 0-forms are zero or central (*i.e.*, singlet with respect to all 1-form connections).

2.4 Dynamical content

General situation can be illustrated by the simple $1d$ example. First-order ordinary differential equations have the following structure

$$\frac{\partial}{\partial t} q_i^{\tilde{\alpha}} = a_i^{\tilde{\alpha}} q_{i+1}^{\tilde{\alpha}} + \dots, \quad i = 0, 1, 2, \dots, \quad (2.24)$$

where $a_i^{\tilde{\alpha}}$ are some coefficients while ellipsis denote nonlinear corrections. If all $a_i^{\tilde{\alpha}}$ are nonzero, Eqs. (2.24) treated perturbatively describe an infinite set of constraints that express all $q_{i+1}^{\tilde{\alpha}}$ via derivatives of $q_0^{\tilde{\alpha}}$. If some coefficient $a_j^{\tilde{\alpha}}$ vanishes, this implies a nontrivial differential equation on $q_0^{\tilde{\alpha}}$, which is of order j at the linearized level. These two options are analogous, respectively, to the off-shell and on-shell cases in the scalar field example of Section 2.2.2. Indeed, using that $1d$ traceless tensors in one dimension are all zero except for C and C_n , the on-shell system (2.17), (2.20) at $d = 1$ implies $\frac{\partial^2}{\partial t \partial t} C = 0$.

In the first-order formulation (in particular, in the Hamiltonian formalism) the initial data problem is set in terms of values of all variables q at given time t_0 . In the general case of $d > 1$ these properties have clear analogues. Nontrivial dynamical fields (*i.e.*, those that are different from *auxiliary fields* expressed via derivatives of the dynamical fields), gauge symmetries and true differential field equations, are classified in terms of the so-called σ_- cohomology [34] that roughly speaking controls zeros among the coefficients analogous to $a_i^{\tilde{\alpha}}$ of the linearized equations. The σ_- cohomology is a perturbative concept that emerges in the linearized analysis with

$$W^\Omega(x) = W_0^\Omega(x) + W_1^\Omega(x), \quad (2.25)$$

where $W_0^\Omega(x)$ is a particular solution of (2.2) and $W_1^\Omega(x)$ is treated as a perturbation. $W_0^\Omega(x)$ is nonzero in a field-theoretical system because, as explained in Section 2.2.1, it should contain a background gravitational field. Linearized equations (2.2)

$$dW_1^\Omega(x) = W_1^\Lambda(x) \frac{\delta G^\Omega}{\delta W^\Lambda} \Big|_{W=W_0} \quad (2.26)$$

can be rewritten in the form (2.16)

$$D_0 W_1^\Omega(x) = 0, \quad (2.27)$$

where D_0 is some differential that squares to zero to fulfill the consistency condition (2.5),

$$D_0^2 = 0. \quad (2.28)$$

Usually, a set of fields W_1 admits a grading G with the spectrum bounded from below, which typically counts a rank of a tensor. Suppose that

$$D_0 = \mathcal{D}_0 + \sigma_- + \sigma_+, \quad (2.29)$$

where

$$[G, \sigma_-] = -\sigma_-, \quad [G, \mathcal{D}_0] = 0 \quad (2.30)$$

and σ_+ is a sum of operators of positive grade. From (2.28) it follows that

$$\sigma_-^2 = 0. \quad (2.31)$$

Provided that σ_- does not differentiate $x^{\mathfrak{n}}$, dynamical content of the system in question is determined by cohomology of σ_- . Namely, as shown in [34] (see also [32]), for p_Ω -forms W^Ω valued in a vector space V , $H^{p+1}(\sigma_-, V)$, $H^p(\sigma_-, V)$ and $H^{p-1}(\sigma_-, V)$ describe, respectively, differential equations, dynamical fields and differential gauge symmetries encoded by equation (2.27). The case with $H^{p+1}(\sigma_-, V) = 0$ is analogous to that of (2.24) with all coefficients $a_i^{\tilde{\alpha}}$ different from zero. Here no differential equations on the dynamical variables are imposed, *i.e.*, equations (2.2) just encode constraints on auxiliary fields. Equations of this type are referred to as *off-shell*. (Let us stress that this definition is true both for linear and non-linear cases: nonlinear equations are off-shell if their linearization is off-shell.) If $H^{p+1}(\sigma_-, V) \neq 0$, unfolded equations (2.2) impose some differential equations on the dynamical fields. Such systems are called *on-shell*.

Degrees of freedom, *i.e.*, variables that determine a (local) solution of equations (2.2) modulo gauge ambiguity, are represented by values of all 0-forms $C^\phi(x_0)$ among $W^\Omega(x_0)$ at any given x_0 which is analogous to t_0 of the $1d$ case. This means in particular that to unfold a field-theoretical system with infinite number of degrees of freedom, an infinite set of 0-forms has to be introduced. In the scalar field example this is the set of 0-forms $C_{n_1 \dots n_k}(x)$.

2.5 Generalized twistor space

Unfolded scalar field system is most conveniently described in terms of generating functions of auxiliary variables y^n

$$C(y|x) = \sum_{k=0}^{\infty} \frac{1}{k!} y^{n_1} \dots y^{n_k} C_{n_1 \dots n_k}(x). \quad (2.32)$$

In the on-shell case, tracelessness of $C_{n_1 \dots n_k}(x)$ is equivalent to the condition that $C(y|x)$ is harmonic in y^n

$$\square_y C(y|x) = 0 \quad (2.33)$$

while the unfolded equations (2.17) take the form

$$(d_x - e^n \frac{\partial}{\partial y^n}) C(y|x) = 0. \quad (2.34)$$

In these terms

$$G = y^n \frac{\partial}{\partial y^n}, \quad D_0 = d, \quad \sigma_+ = 0, \quad \sigma_- = -e^n \frac{\partial}{\partial y^n}. \quad (2.35)$$

In the off-shell case, σ_- acts as exterior differential on polynomials. Hence, the only nontrivial σ_- -cohomology is $H^0(\sigma_-)$. This tells us that scalar $C(x)$ is the only dynamical field and that it is not restricted by dynamical equations. In the on-shell case, $H^1(\sigma_-)$ is one dimensional in agreement with the fact that the on-shell unfolded system imposes only one equation on the scalar field, namely Klein-Gordon equation, which belongs to the following part of the unfolded equations

$$e_m^m \frac{\partial^2}{\partial d x^m \partial y_m} \left(d_x - e^a \frac{\partial}{\partial y^a} \right) C(y|x) \Big|_{y=0} = 0, \quad (2.36)$$

where the vector field e_m^m is the inverse to the (co)vielbein 1-form e_m^m .

In field-theoretical systems with infinite degrees of freedom the label Ω , enumerating differential forms in the unfolded equations (2.2), usually refers to appropriate functional spaces, *i.e.*, $W^\Omega(X)$ can be represented as a set $W^i(Y|X)$ of functions of some auxiliary variables Y (i labels different functions).

The variables Y are analogous to the twistor coordinates in the twistor theory. Following [35] we interpret Y as coordinates of a generalized twistor space¹ \mathbf{T} . X are space-time coordinates of space \mathbf{M} . Together, (Y, X) are interpreted as coordinates of the correspondence space \mathbf{C} . More precisely, \mathbf{C} is a fiber bundle with \mathbf{M} as base manifold and \mathbf{T} as fibers.

In this setup, the unfolded equation (2.2) encodes a generalized Penrose transform [37, 36] expressed by the diagram

$$\begin{array}{ccc} & \mathbf{C} & \\ \eta \swarrow & & \searrow \nu \\ \mathbf{M} & & \mathbf{T} \end{array} . \quad (2.37)$$

Indeed, it reconstructs (locally) general solution $W^i(Y|X)$ of the unfolded equations in terms of arbitrary functions $W^i(Y|X_0)$ on the twistor space \mathbf{T} . Via restriction of a p -form $W^i(Y|X)$ to dynamical fields associated with $H^p(\sigma_-)$ this gives a solution of the dynamical equations associated with the $H^{p+1}(\sigma_-)$ part of the unfolded equations.

As discussed in more detail in Section 4, in lower dimensions $d = 3, 4$, the on-shell condition (2.33) is conveniently resolved in terms of unrestricted functions of spinor (twistor) variables. This greatly simplifies analysis of the respective unfolded on-shell systems.

As discussed in Section 3, holographic duality relates differently looking dynamical systems in different space-times (coordinates X) that have the same twistor space. From this perspective unfolded equations perform a generalized Penrose transform from the same twistor space \mathbf{T} to one or another space-time \mathbf{M} .

¹In HS theories, \mathbf{T} may or may not have precise geometric meaning of some twistor space in the twistor theory [36]. Nevertheless, abusing terminology, in this paper we call it twistor space.

2.6 Properties

Unfolded formulation has a number of useful and important properties discussed in some more detail, *e.g.*, in [33]. In particular, unfolded equations possess manifest gauge and diffeomorphism invariance due to exterior algebra formalism. As such, they are perfectly suited for the study of gauge invariant theories in the framework of gravity like, *e.g.*, HS gauge theories. The following properties of unfolded dynamics are most relevant to the analysis of holography.

The method is universal. Any dynamical system can be unfolded. This is analogous to the fact that any system of ordinary differential equations admits a first-order formulation.

Indeed, let $w = e_0^a P_a + \frac{1}{2}\omega_0^{ab} M_{ab}$ be a vacuum connection valued in some space-time symmetry algebra h . Let a field $C^{(0)}(X)$ satisfy some dynamical equations to be unfolded. Consider first the case where $C^{(0)}(X)$ is a 0-form. One starts by writing the equation $D_0^L C^{(0)} = e_0^a C_a^{(1)}$, where D_0^L is the Lorentz covariant derivative and the field $C_a^{(1)}$ is auxiliary. Next, one checks whether the original field equations for $C^{(0)}$ impose restrictions on the first derivatives of $C^{(0)}$. A part of $D_{0\underline{m}}^L C^{(0)}$, and hence $C^{(1)}$, may vanish on the mass-shell (*e.g.* for Dirac equation). The remaining non-zero auxiliary fields $C^{(1)}$ parameterize on-mass-shell nontrivial components of first derivatives. One proceeds by writing analogous equation for the first-level auxiliary fields $D_0^L C_a^{(1)} = e_0^b C_{a,b}^{(2)}$ where the new fields $C_{a,b}^{(2)}$ parameterize second derivatives of $C^{(0)}$. Again, one checks, taking into account Bianchi identities, which components of the second level fields $C_{a,b}^{(2)}$ remain nonzero if the original equations of motion are satisfied. This process continues indefinitely, leading to a chain of equations on the chain of fields $C_{a_1, a_2, \dots, a_m}^{(m)}$ ($m \in \mathbb{N}$) parameterizing all on-mass-shell nontrivial derivatives of the original dynamical field. If one starts with some gauge field, analogous analysis determines a form of shift gauge transformations that subtract extra field components to be introduced to describe a system in terms of differential forms. (For instance, local Lorentz symmetry in Cartan formulation of gravity appears this way as the shift symmetry that removes extra components of the vielbein 1-form compared to the metric tensor.) By construction, this leads to a particular unfolded system. The correspondence between $p \geq 1$ forms and gauge symmetries in the unfolded dynamics approach uncovers pattern of local and global symmetries associated with a given gauge field. In particular, the pattern of the linearized $4d$ HS algebras was deduced this way in [38]. These results were then used in [39, 40] to find infinite-dimensional non-Abelian HS algebras that underly the nonlinear $4d$ HS theories.

In the topologically trivial situation, degrees of freedom are carried by 0-forms at any space-time point $X = X_0$. Indeed, by virtue of Poincaré lemma, unfolded equations express all exterior derivatives in terms of the values of fields themselves modulo exact forms that can be gauged away by the gauge transformation (2.9). What is left is the “constant part” of 0-forms. In terms of functions of twistor variables Y like $C^i(Y|X)$ this means that dynamics is entirely encoded by 0-forms on the twistor space, *i.e.*, $C^i(Y|X_0)$ at any X_0 .

3 Unfolding and holographic duality

Unfolded formulation unifies various dual versions of one and the same system. This concerns both duality between systems in the same space-time and holographic duality between theories in space-times of different dimensions.

In the former case, duality results from the ambiguity in which fields are chosen to be dynamical or auxiliary, the nomenclature governed by the choice of the grading G and σ_- . Different gradings lead to different interpretations of the same unfolded system in terms of different dynamical fields that satisfy seemingly unrelated differential equations. The key point is that if two dynamical systems give rise to the same unfolded system (more precisely, belong to the same projective system [33]), they are equivalent.

Holographic duality rests on the striking feature that a universal unfolded system may admit different space-time interpretations. In some sense, space-time dependence in such systems is auxiliary as was first noted in the context of HS dynamics in [41]. True dynamics is hidden in the twistor sector of the auxiliary variables Y .

Indeed, in a universal unfolded system dynamics is entirely encoded by the function $G^\Omega(W)$ independently of the original space-time picture. In particular, unfolded formulation allows one to extend space-time without changing dynamics simply by letting the differential d and differential forms W^Φ to live in a larger space

$$d = dX^n \frac{\partial}{\partial X^n} \rightarrow \tilde{d} = dX^n \frac{\partial}{\partial X^n} + d\hat{X}^{\hat{n}} \frac{\partial}{\partial \hat{X}^{\hat{n}}}, \quad dX^n W_n \rightarrow dX^n W_n + d\hat{X}^{\hat{n}} \hat{W}_{\hat{n}}, \quad (3.1)$$

where $\hat{X}^{\hat{n}}$ are some additional coordinates. For a universal unfolded system such substitution neither spoils consistency nor affects local dynamics still determined by the 0-forms at any point of (any) space-time. Indeed, the unfolded system in the X space remains a subsystem of that in the enlarged space while additional equations reconstruct dependence on $\hat{X}^{\hat{n}}$ in terms of solutions of the original system (of course, this consideration is local).

On the other hand, as emphasized in [42], the role of coordinates is that they help to visualize physical local events via a physical processes. A particular space-time interpretation of a universal unfolded system, e.g, whether a system is on-shell or off-shell, depends not only on $G^\Omega(W)$ but, in the first place, on a chosen space-time M^d and vacuum solution $W_0(X)$. Dynamical interpretation may be different for different space-times because σ_- cohomology depends on space-time dimension via rank of the vielbein 1-form. In particular, the on-shell HS theory in AdS_4 will be shown in Section 9 to be dual to an off-shell $3d$ conformal system. This implies that the $3d$ “dynamical” boundary fields are not restricted by any differential field equations. Among other things this property makes it possible to identify $3d$ dynamical fields with unrestricted boundary values of the on-shell bulk fields.

Important point considered in more detail in Section 13 is that, in unfolded formulation, a nontrivial conserved charge or gauge invariant generating functional for correlators is represented as integral of some d -closed form [26]. As a result, correlators in boundary conformal systems are represented by integrals over a space larger than space-time where conformal fields live. It can be either the bulk space as in the standard AdS/CFT treatment

or some twistor space. In any case, from this perspective the nonlinear off-shell system at the boundary is represented by a nonlinear on-shell system in a larger space².

To summarize, two unfolded systems in different space-times are equivalent (dual) provided that they have the same unfolded form. Generally, a most straightforward way to establish holographic duality between two theories is to unfold both of them to see whether the operators Q (2.7) of their unfolded formulations coincide. Other way around, given unfolded system generates a class of holographically dual theories in different dimensions. It should be stressed that, being simple in terms of unfolded dynamics and the corresponding twistor space \mathbf{T} , holographic duality in usual space-time terms may be very complicated requiring solution of at least one of the two unfolded systems which is equivalent to a nonlinear integral transform.

4 Free massless fields in AdS_4

In this section, we remind the reader unfolded formulation of free massless fields of all spins in AdS_4 obtained originally in [27]. It is based on the frame-like approach to HS gauge fields [43, 38] where a $4d$ spin $s \geq 1$ massless field is described by the set of 1-forms

$$\omega_{\alpha_1 \dots \alpha_k, \dot{\alpha}_1 \dots \dot{\alpha}_l} = dx^{\underline{n}} \omega_{\underline{n} \alpha_1 \dots \alpha_k, \dot{\alpha}_1 \dots \dot{\alpha}_l}, \quad k + l = 2(s - 1), \quad (4.1)$$

where $\alpha, \beta \dots = 1, 2$ and $\dot{\alpha}, \dot{\beta} \dots = 1, 2$ are two-component spinor indices. The HS gauge fields are totally symmetric with respect to each type of spinor indices and obey the reality conditions $\overline{\omega_{\alpha_1 \dots \alpha_k, \dot{\beta}_1 \dots \dot{\beta}_l}} = \omega_{\beta_1 \dots \beta_l, \dot{\alpha}_1 \dots \dot{\alpha}_k}$. For a given s , this set is equivalent to the real 1-form $\omega_{A_1 \dots A_{2(s-1)}}$ symmetric in the Majorana indices $A = 1, 2, 3, 4$. As such it carries an irreducible module of $sp(4, \mathbb{R}) \sim o(3, 2)$.

AdS_4 is described by the Lorentz connection $\omega^{\alpha\beta}$, $\bar{\omega}^{\dot{\alpha}\dot{\beta}}$ and vierbein $e^{\alpha\dot{\alpha}}$. Altogether they form the $sp(4, \mathbb{R})$ connection $w^{AB} = w^{BA}$ that satisfies the $sp(4, \mathbb{R})$ zero-curvature conditions

$$R^{AB} = 0, \quad R^{AB} = dw^{AB} + w^{AC} \wedge w_C^B, \quad (4.2)$$

where indices are raised and lowered by a $sp(4, \mathbb{R})$ invariant form $C_{AB} = -C_{BA}$

$$A_B = A^A C_{AB}, \quad A^A = C^{AB} A_B, \quad C_{AC} C^{BC} = \delta_A^B. \quad (4.3)$$

²There is a subtlety related to this point to be taken into account in the holography context. Perturbatively, every nonlinear off-shell system is equivalent to some linear system by virtue of a perturbatively local nonlinear field redefinition. Indeed, a system is off-shell in the case where the operator σ_- is a kind of nondegenerate. Writing schematically $R_n = dw_n + \sigma_- w_{n+1} + \dots$, where ellipsis denote lower-grade and/or nonlinear terms one observes that the latter can all be absorbed into nonlinear field redefinitions of w_k with $k = 1, 2, \dots$ modulo σ_- exact terms that are pure gauge with respect to some shift symmetries. Although such a field redefinition destroys the structure of unfolded equations, the resulting system has linear form. This should be compared to on-shell nonlinear systems where not all nonlinear terms can be removed by such a field redefinition. In particular, the l.h.s.s of field equations, associated with the respective σ_- cohomology, cannot be linearized in a general nonlinear on-shell system.

In terms of Lorentz components $w^{AB} = (\omega^{\alpha\beta}, \bar{\omega}^{\dot{\alpha}\dot{\beta}}, \lambda e^{\alpha\dot{\beta}}, \lambda e^{\beta\dot{\alpha}})$, where λ^{-1} is the AdS_4 radius, the AdS_4 equations (4.2) read as

$$R_{\alpha\beta} = 0, \quad \bar{R}_{\dot{\alpha}\dot{\beta}} = 0, \quad R_{\alpha\dot{\alpha}} = 0, \quad (4.4)$$

where

$$R_{\alpha\beta} = d\omega_{\alpha\beta} + \omega_{\alpha}{}^{\gamma} \wedge \omega_{\beta\gamma} + \lambda^2 e_{\alpha}{}^{\dot{\delta}} \wedge e_{\beta\dot{\delta}}, \quad (4.5)$$

$$\bar{R}_{\dot{\alpha}\dot{\beta}} = d\bar{\omega}_{\dot{\alpha}\dot{\beta}} + \bar{\omega}_{\dot{\alpha}}{}^{\dot{\gamma}} \wedge \bar{\omega}_{\dot{\beta}\dot{\gamma}} + \lambda^2 e^{\gamma}{}_{\dot{\alpha}} \wedge e_{\gamma\dot{\beta}},$$

$$R_{\alpha\dot{\beta}} = de_{\alpha\dot{\beta}} + \omega_{\alpha}{}^{\gamma} \wedge e_{\gamma\dot{\beta}} + \bar{\omega}_{\dot{\beta}}{}^{\dot{\delta}} \wedge e_{\alpha\dot{\delta}}. \quad (4.6)$$

(Two-component indices are raised and lowered as in (4.3) with $\epsilon_{\alpha\beta}$ or $\epsilon_{\dot{\alpha}\dot{\beta}}$ instead of C_{AB} .)
Unfolded equations of motion of a spin- s massless field are [27]

$$R_{\alpha_1 \dots \alpha_n, \dot{\alpha}_1 \dots \dot{\alpha}_m} = \eta \delta_n^0 \bar{H}^{\dot{\alpha}_{2s-1} \dot{\alpha}_{2s}} \bar{C}_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} + \bar{\eta} \delta_m^0 H^{\alpha_{2s-1} \alpha_{2s}} C_{\alpha_1 \dots \alpha_{2s}}, \quad n + m = 2(s-1) \quad (4.7)$$

and

$$D^{tw} C_{\alpha_1 \dots \alpha_n, \dot{\alpha}_1 \dots \dot{\alpha}_m} = 0, \quad n - m = 2s, \quad D^{tw} \bar{C}_{\alpha_1 \dots \alpha_n, \dot{\alpha}_1 \dots \dot{\alpha}_m} = 0, \quad m - n = 2s. \quad (4.8)$$

Here the HS field strength and twisted adjoint covariant derivative have the form

$$R_{\alpha_1 \dots \alpha_n, \dot{\alpha}_1 \dots \dot{\alpha}_m} := D^L \omega_{\alpha_1 \dots \alpha_n, \dot{\alpha}_1 \dots \dot{\alpha}_m} + n \lambda e_{\alpha_1}{}^{\dot{\alpha}_{m+1}} \wedge \omega_{\alpha_2 \dots \alpha_n, \dot{\alpha}_1 \dots \dot{\alpha}_{m+1}} + m \lambda e^{\alpha_{n+1}}{}_{\dot{\alpha}_1} \wedge \omega_{\alpha_1 \dots \alpha_{n+1}, \dot{\alpha}_2 \dots \dot{\alpha}_m}, \quad (4.9)$$

$$D^{tw} C_{\alpha_1 \dots \alpha_n, \dot{\alpha}_1 \dots \dot{\alpha}_m} := D^L C_{\alpha_1 \dots \alpha_n, \dot{\alpha}_1 \dots \dot{\alpha}_m} - \frac{i}{2} \lambda (e^{\gamma\dot{\delta}} C_{\alpha_1 \dots \alpha_n \gamma, \dot{\alpha}_1 \dots \dot{\alpha}_m \dot{\delta}} - n m e_{\alpha_1 \dot{\alpha}_1} C_{\alpha_2 \dots \alpha_n, \dot{\alpha}_2 \dots \dot{\alpha}_m}), \quad (4.10)$$

where the indices α and $\dot{\alpha}$ are (separately) symmetrized and Lorentz covariant derivative D^L is

$$D^L \psi_{\alpha} := d\psi_{\alpha} + \omega_{\alpha}{}^{\beta} \psi_{\beta}, \quad D^L \bar{\psi}_{\dot{\alpha}} := d\bar{\psi}_{\dot{\alpha}} + \bar{\omega}_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\psi}_{\dot{\beta}}. \quad (4.11)$$

$H^{\alpha\beta} = H^{\beta\alpha}$ and $\bar{H}^{\dot{\alpha}\dot{\beta}} = \bar{H}^{\dot{\beta}\dot{\alpha}}$ are the basis 2-forms

$$H^{\alpha\beta} := e^{\alpha}{}_{\dot{\alpha}} \wedge e^{\beta\dot{\alpha}}, \quad \bar{H}^{\dot{\alpha}\dot{\beta}} := e_{\alpha}{}^{\dot{\alpha}} \wedge e^{\alpha\dot{\beta}}. \quad (4.12)$$

The phase parameters η and $\bar{\eta}$ ($\eta\bar{\eta} = 1$) are introduced for the future convenience. Although at the linearized level they can be absorbed into redefinition of mutually conjugated C and \bar{C} , they become nontrivial beyond the linearized approximation [27] where the cases of $\eta = 1$ and $\eta = i$ correspond to two parity symmetric types of HS theories often referred to as, respectively, A and B model[22].

Formulae are simplified in terms of generating functions

$$A(y, \bar{y} | x) = i \sum_{n,m=0}^{\infty} \frac{1}{n!m!} y_{\alpha_1} \dots y_{\alpha_n} \bar{y}_{\dot{\beta}_1} \dots \bar{y}_{\dot{\beta}_m} A^{\alpha_1 \dots \alpha_n, \dot{\beta}_1 \dots \dot{\beta}_m}(x) \quad (4.13)$$

with $A = \omega, C, \overline{C}, R$ etc. In particular,

$$R(y, \bar{y}|x) = D^{ad}\omega(y, \bar{y}|x) = D^L\omega(y, \bar{y}|x) - \lambda e^{\alpha\dot{\beta}} \left(y_\alpha \frac{\partial}{\partial \bar{y}^\beta} + \frac{\partial}{\partial y^\alpha} \bar{y}_{\dot{\beta}} \right) \omega(y, \bar{y}|x), \quad (4.14)$$

$$D^{tw}C(y, \bar{y}|x) = D^LC(y, \bar{y}|x) + \frac{i}{2} \lambda e^{\alpha\dot{\beta}} \left(y_\alpha \bar{y}_{\dot{\beta}} - \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta} \right) C(y, \bar{y}|x), \quad (4.15)$$

$$D^LA(y, \bar{y}|x) = dA(y, \bar{y}|x) - \left(\omega^{\alpha\beta} y_\alpha \frac{\partial}{\partial y^\beta} + \bar{\omega}^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \frac{\partial}{\partial \bar{y}^\beta} \right) A(y, \bar{y}|x). \quad (4.16)$$

As a consequence of the AdS_4 zero-curvature equation (4.2), the covariant derivatives D^{ad} and D^{tw} are flat, *i.e.*,

$$(D^{ad})^2 = (D^{tw})^2 = 0.$$

These conditions imply consistency of equations (4.7) and (4.8) (*i.e.*, compatibility with $d^2 = 0$) and gauge invariance of the field strength (4.14) (and hence free equations (4.7)) under Abelian HS gauge transformations

$$\delta\omega(y, \bar{y}|x) = D^{ad}\epsilon(y, \bar{y}|x). \quad (4.17)$$

It is important that consistency of the equations is not spoiled by the C -dependent terms in (4.7). As explained in [33], this means that the latter correspond to Chevalley-Eilenberg cohomology of $sp(4, \mathbb{R})$ with coefficients in the corresponding infinite-dimensional module.

In Eqs. (4.7), (4.8), a spin s field is described by the set of gauge 1-forms $\omega^{\alpha_1 \dots \alpha_n, \dot{\beta}_1 \dots \dot{\beta}_m}(x)$ with $n + m = 2(s - 1)$ (for $s \geq 1$) and 0-forms $C^{\alpha_1 \dots \alpha_n, \dot{\beta}_1 \dots \dot{\beta}_m}(x)$ with $n - m = 2s$ along with their conjugates $\overline{C}^{\alpha_1 \dots \alpha_n, \dot{\beta}_1 \dots \dot{\beta}_m}(x)$ with $m - n = 2s$. Indeed, it is easy to see that the field equations (4.7) and (4.8) for such a set of fields form a subsystem for any s .

For example, a spin zero field is described by a set of multispinors $C^{\alpha_1 \dots \alpha_n, \dot{\beta}_1 \dots \dot{\beta}_n}(x)$ which is equivalent to the set of $4d$ symmetric tensors $C_{m_1 \dots m_n}(x)$ satisfying the tracelessness condition (2.20). This property extends to all spins in the sense that all multispinors $C^{\alpha_1 \dots \alpha_n, \dot{\beta}_1 \dots \dot{\beta}_m}(x)$ describe traceless Lorentz tensors. This follows from the Penrose formula that any $p_{\alpha\dot{\alpha}} = p_\alpha \bar{p}_{\dot{\alpha}}$ is null [37]. Indeed, unfolded equations just express $p_{\alpha\dot{\alpha}} \sim \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}$ via $p_\alpha \bar{p}_{\dot{\alpha}}$ with $p_\alpha \sim \frac{\partial}{\partial y^\alpha}$, $\bar{p}_{\dot{\alpha}} \sim \frac{\partial}{\partial \bar{y}^{\dot{\alpha}}}$ (modulo mass-like terms proportional to $\lambda^2 y^\alpha y^{\dot{\alpha}}$ necessary in AdS background).

Dynamical massless fields are

- $C(x)$ and $\overline{C}(x)$ for two spin 0 fields,
- $C_\alpha(x)$ and $\overline{C}_{\dot{\alpha}}(x)$ for a massless spin 1/2 field,
- $\omega_{\alpha_1 \dots \alpha_{s-1}, \dot{\alpha}_1 \dots \dot{\alpha}_{s-1}}(x)$ for an integer spin $s \geq 1$ massless field,
- $\omega_{\alpha_1 \dots \alpha_{s-3/2}, \dot{\alpha}_1 \dots \dot{\alpha}_{s-1/2}}(x)$ and its complex conjugate $\omega_{\alpha_1 \dots \alpha_{s-1/2}, \dot{\alpha}_1 \dots \dot{\alpha}_{s-3/2}}(x)$ for a half-integer spin $s \geq 3/2$ massless field.

All other fields are auxiliary, being expressed via derivatives of the dynamical massless fields by Eqs. (4.7), (4.8).

Pattern of the unfolded massless field equations is expressed by the so called Central On-Shell Theorem [27] stating that Eqs. (4.7), (4.8) express all auxiliary fields in terms of derivatives of the dynamical fields, imposing on the latter massless field equations equivalent to those of Fronsdal [44] and Fang and Fronsdal [45].

More in detail, the meaning of Eqs. (4.7), (4.8) is as follows. Eqs. (4.8) are independent from (4.7) for spins $s = 0$ and $s = \frac{1}{2}$ and partially independent for $s = 1$ but become consequences of (4.7) for $s > 1$. Eqs. (4.7) express the holomorphic and antiholomorphic components of spin $s \geq 1$ 0-forms $C(y, \bar{y}|x)$ via derivatives of the gauge 1-forms $\omega(y, \bar{y}|x)$. This identifies the spin $s \geq 1$ holomorphic and antiholomorphic components of the 0-forms $C(y, \bar{y}|x)$ with the Maxwell tensor, on-shell Rarita-Schwinger curvature, Weyl tensor and their HS counterparts considered already in the seminal works [46, 37]. In addition, Eqs. (4.7) impose usual (second-order for bosons and first-order for fermions) field equations on the spin $s > 1$ massless gauge fields so that Eqs. (4.8) become their consequences by virtue of Bianchi identities. Dynamical equations for spins $s \leq 1$ are contained in Eqs. (4.8).

Although the system (4.7), (4.8) is consistent at the free field level, its nonlinear extension requires doubling of fields [27, 1, 2]. This is achieved by introducing the fields

$$\omega^{ii}(y, \bar{y}|x), \quad C^{i1-i}(y, \bar{y}|x), \quad i = 0, 1$$

such that $\omega^{ii}(y, \bar{y}|x)$ are selfconjugated, while $C^{01}(y, \bar{y}|x)$ and $C^{10}(y, \bar{y} | x)$ are conjugated to one another,

$$\overline{\omega^{ii}(y, \bar{y}|x)} = \omega^{ii}(\bar{y}, y|x), \quad \overline{C^{i1-i}(y, \bar{y}|x)} = C^{1-i i}(\bar{y}, y|x).$$

The unfolded system for the doubled set of fields is

$$R^{ii}(y, \bar{y} | x) = \eta \bar{H}^{\dot{\alpha}\dot{\beta}} \frac{\partial^2}{\partial \bar{y}^{\dot{\alpha}} \partial \bar{y}^{\dot{\beta}}} C^{1-i i}(0, \bar{y} | x) + \bar{\eta} H^{\alpha\beta} \frac{\partial^2}{\partial y^{\alpha} \partial y^{\beta}} C^{i1-i}(y, 0 | x), \quad (4.18)$$

$$D^{tw} C^{i1-i}(y, \bar{y} | x) = 0. \quad (4.19)$$

Note that now all components of the expansions (4.13) of $C^{i1-i}(y, \bar{y} | x)$ contribute to Eqs. (4.18), (4.19), while in Eqs. (4.7), (4.8) with the single HS 1-form $\omega(y, \bar{y})$ only those components of $C(y, \bar{y})$ ($\bar{C}(y, \bar{y})$) contributed, that carried at least as many $y(\bar{y})$ as $\bar{y}(y)$.

In the standard formulation of the 4d nonlinear HS gauge theory [2, 3], the field doubling results from the dependence on the Klein operators k and \bar{k} that have the properties

$$kw^{\alpha} = -w^{\alpha}k, \quad k\bar{w}^{\dot{\alpha}} = \bar{w}^{\dot{\alpha}}k, \quad \bar{k}w^{\alpha} = w^{\alpha}\bar{k}, \quad \bar{k}\bar{w}^{\dot{\alpha}} = -\bar{w}^{\dot{\alpha}}\bar{k}, \quad k^2 = \bar{k}^2 = 1, \quad k\bar{k} = \bar{k}k, \quad (4.20)$$

where $w^{\alpha} = y^{\alpha}$, $\bar{w}^{\dot{\alpha}} = \bar{y}^{\dot{\alpha}}$. All fields are packed into 1-forms

$$\omega(y, \bar{y}; k, \bar{k} | x) = \sum_{ij=0,1} (k)^i (\bar{k})^j \omega^{ij}(y, \bar{y} | x)$$

and 0-forms

$$C(y, \bar{y}; k, \bar{k} | x) = \sum_{ij=0,1} (k)^i (\bar{k})^j C^{ij}(y, \bar{y} | x).$$

Now both adjoint and twisted adjoint covariant derivatives result from different sectors of the adjoint covariant derivative in the Weyl algebra extended by the Klein operators.

Massless fields are those with

$$\omega(y, \bar{y}; -k, -\bar{k} | x) = \omega(y, \bar{y}; k, \bar{k} | x), \quad C(y, \bar{y}; -k, -\bar{k} | x) = -C(y, \bar{y}; k, \bar{k} | x).$$

The fields with the opposite oddnesses in the Klein operators are topological, carrying at most a finite number of degrees of freedom per an irreducible subsystem [47].

Truncating out fermions, it is possible to consider a system with bosonic fields in which every integer spin appears once. This is achieved via projection of the theory with the help of projectors

$$\Pi_{\pm} = \frac{1}{2}(1 \pm k\bar{k}), \quad (4.21)$$

which are central in the bosonic HS theory. HS gauge fields and Weyl 0-forms of the bosonic theory are $\omega_{\pm} = \frac{1}{2}(\omega^{00} \pm \omega^{11})$ and $C_{\pm} = \frac{1}{2}(C^{01} \pm C^{10})$. Bosonic HS theories can be further truncated to the minimal system that only contains even spins [3].

5 Conserved currents and massless equations

As observed in [48, 9, 33], conformal invariant massless equations are naturally formulated in the spaces \mathcal{M}_M with matrix coordinates $X^{AB} = X^{BA}$ ($A, B = 1, \dots, M$). More precisely, 3d Minkowski space-time coincides with \mathcal{M}_2 while 4d Minkowski space-time is a subspace of the ten-dimensional space \mathcal{M}_4 . In the both cases unfolded massless field equations are

$$dX^{AB} \left(\frac{\partial}{\partial X^{AB}} \pm \frac{\partial^2}{\partial Y^A \partial Y^B} \right) C_{\pm}(Y|X) = 0, \quad (5.1)$$

where \pm is introduced for the future convenience.

In [21], Eq. (5.1) was extended to so-called rank- r unfolded equations

$$dX^{AB} \left(\frac{\partial}{\partial X^{AB}} \pm \eta^{ij} \frac{\partial^2}{\partial Y^{iA} \partial Y^{jB}} \right) C_{\pm}^r(Y|X) = 0, \quad (5.2)$$

where $i, j = 1, \dots, r$ and $\eta^{ij} = \eta^{ji}$ is some nondegenerate metric. Higher-rank systems inherit all symmetries of the underlying lower-rank system simply because they correspond to the tensor product of the lower-rank representation. In particular, higher-rank systems are conformal once the underlying lower-rank systems were. In the basis where η^{ij} is diagonal, the higher-rank equation (5.2) is satisfied by the product of rank-one fields

$$C^r(Y_i|X) = C_1(Y_1|X) C_2(Y_2|X) \dots C_r(Y_r|X). \quad (5.3)$$

A rank- r system in \mathcal{M}_M can be further extended to the rank-one system (5.1) in the larger space \mathcal{M}_{rM} with coordinates X_{ij}^{AB} via reinterpretation of the twistor coordinates

$$Y_i^A \rightarrow Y^{\tilde{A}}, \quad \tilde{A} = 1, \dots, rM. \quad (5.4)$$

The diagonal embedding of \mathcal{M}_M into \mathcal{M}_{rM} is

$$X_{11}^{AB} = X_{22}^{AB} = \dots = X_{rr}^{AB} = X^{AB}. \quad (5.5)$$

In particular, the $M = 2$ rank-two system extends to the $M = 4$ rank-one system. Group theoretically, this provides a realization of the Flato and Fronsdal theorem [12] that relates tensor products of $3d$ conformal fields (singletons) to infinite towers of $4d$ massless fields of all spins. The key fact underlying AdS_4/CFT_3 holographic duality in HS theories is that, as shown in [21], rank-two systems just describe conserved currents. Direct identification of $3d$ conserved conformal currents with $4d$ massless fields provides an example of holographic duality via unfolded dynamics.

Let us recall following [25] how rank-two equations give rise to conserved currents. The rank-two equation can be rewritten in the form

$$\left\{ \frac{\partial}{\partial X^{AB}} - \frac{\partial^2}{\partial Y^{(A} \partial U^{B)}} \right\} T(U, Y|X) = 0, \quad (5.6)$$

where $T(U, Y|X)$ will be called generalized stress tensor. In particular, Eq. (5.6) is obeyed by the bilinear substitution

$$T(U, Y|X) = \sum_{i=1}^N C_{+i}(Y - U|X) C_{-i}(U + Y|X), \quad (5.7)$$

where $C_{\pm i}(Y|X)$ obey (5.1). Rank-two fields can be interpreted as bilocal fields in the twistor space. Being seemingly similar to the bilocal space-time formalism of [49, 50, 51], the twistorial bilocal formalism is in many respects more efficient.

Since Eq. (5.6) has unfolded form, its dynamical pattern can be analyzed with the help of σ_- -cohomology techniques with

$$\sigma_- = -dX^{AB} \frac{\partial^2}{\partial Y^A \partial U^B}. \quad (5.8)$$

The result is that dynamical currents (primaries), that belong to $H^0(\sigma_-)$, are [21]

$$J(U|X) = T(U, 0|X), \quad \tilde{J}(Y|X) = T(0, Y|X), \quad (5.9)$$

$$J^{asym}(U, Y|X) = (U^A Y^B - U^B Y^A) \left(\frac{\partial^2}{\partial U^A \partial Y^B} T(U, Y|X) \Big|_{U^A=Y^A=0} \right). \quad (5.10)$$

In the $3d$ case of $M = 2$ where $A, B \rightarrow \alpha, \beta$, $J(U|X)$ generates $3d$ currents of all integer and half-integer spins

$$J(U|X) = \sum_{2s=0}^{\infty} U^{\alpha_1} \dots U^{\alpha_{2s}} J_{\alpha_1 \dots \alpha_{2s}}(X) \quad (5.11)$$

and

$$J^{asym}(U, Y|X) = U_\alpha Y^\alpha J^{asym}(X). \quad (5.12)$$

Differential equations, which follow from Eq. (5.6), are associated with $H^1(\sigma_-)$ found in [21] for general M . For $M = 2$, the structure of $H^1(\sigma_-)$ is greatly simplified so that the resulting field equations amount solely to the conventional conservation condition

$$\frac{\partial}{\partial X^{\alpha\beta}} \frac{\partial^2}{\partial U_\alpha \partial U_\beta} J(U|X) = 0, \quad \frac{\partial}{\partial X^{\alpha\beta}} \frac{\partial^2}{\partial Y_\alpha \partial Y_\beta} \tilde{J}(Y|X) = 0. \quad (5.13)$$

To define conserved charges, it is convenient to Fourier transform $T(U, Y|X)$ to

$$\tilde{T}(W, Y|X) = (2\pi)^{-M/2} \int_{\mathbb{R}^M} d^M U \exp(-i W_C U^C) T(U, Y|X), \quad (5.14)$$

which satisfies the following *current equation*

$$\left(\frac{\partial}{\partial X^{AB}} + i W_{(A} \frac{\partial}{\partial Y^{B)}} \right) \tilde{T}(W, Y|X) = 0. \quad (5.15)$$

The key fact is that a $2M$ -form

$$\Omega(T) = \left(dW_A \wedge \left(i W_B dX^{AB} - dY^A \right) \right)^M \tilde{T}(W, Y|X) \quad (5.16)$$

is closed in $M_M \times \mathbb{R}^M(W_B) \times \mathbb{C}^M(Y^A)$ provided that $\tilde{T}(W, Y|X)$ obeys (5.15).

Indeed, from (5.15) and (5.16) it follows that

$$\begin{aligned} & \left(dW_A \frac{\partial}{\partial W_A} + dX^{AB} \frac{\partial}{\partial X^{AB}} + dY^A \frac{\partial}{\partial Y^A} \right) \wedge \Omega^{2M}(T(W, Y|X)) = \\ & = \left(dW_A \frac{\partial}{\partial W_A} - \left(i W_B dX^{AB} - dY^A \right) \frac{\partial}{\partial Y^A} \right) \wedge \Omega^{2M}(T(W, Y|X)) = 0. \end{aligned}$$

As a result, the charge

$$q = q(T) = \int_{\Sigma^{2M}} \Omega^{2M}(T) \quad (5.17)$$

is independent of local variations of a $2M$ -dimensional integration surface Σ^{2M} on solutions of (5.6). In particular, for functions that decrease fast enough at space infinity, it is independent of the time parameter in \mathcal{M}_M , hence being conserved.

A remarkable output of this construction [25] is that it makes it possible to express conserved charges as integrals over the twistor space \mathbf{T} at any point of space-time.

Since (5.15) is a first-order linear PDE system, its solutions form a commutative algebra \mathcal{R} , *i.e.*, a linear combination of products of any regular solutions of (5.15) is also a solution. \mathcal{R} is the algebra of functions η of the form

$$\eta(W, Y|X) = \varepsilon(W_A, Y^C - i X^{CB} W_B) \quad (5.18)$$

with regular $\varepsilon(W, Y)$. As a result, Eq. (5.16) generates conserved currents for $\tilde{T}_\eta(W, Y | X)$ of the form

$$\tilde{T}_\eta(W, Y | X) = \eta(W, Y | X) \tilde{T}(W, Y | X), \quad (5.19)$$

where $\eta(W, Y | X)$ (5.18) is a polynomial representing a parameter of global HS symmetry. The charges $q(\tilde{T}_\eta)$ with various $\eta(W, Y | X)$ generate the full set of conformal HS conserved charges. In particular, at $M = 2$, formula (5.7) generates all conserved charges for free $3d$ massless fields.

6 Nonlinear HS equations in AdS_4

In this section, we first recall standard formulation of nonlinear $4d$ HS equations of [2] and then extend it to a larger space with spinor coordinates. In the sequel, wedge products are implicit.

6.1 Standard formulation

The key element of the construction of [2] is the doubling of auxiliary Majorana spinor variables Y_A in the HS 1-forms and 0-forms

$$\omega(Y; \mathcal{K} | x) \longrightarrow W(Z; Y; \mathcal{K} | x), \quad C(Y; \mathcal{K} | x) \longrightarrow B(Z; Y; \mathcal{K} | x) \quad (6.1)$$

supplemented with equations which determine dependence on the additional variables Z_A in terms of “initial data”

$$\omega(Y; \mathcal{K} | x) = W(0; Y; \mathcal{K} | x), \quad C(Y; \mathcal{K} | x) = B(0; Y; \mathcal{K} | x). \quad (6.2)$$

To this end, we introduce a spinor field $S_A(Z; Y; \mathcal{K} | x)$ that carries only pure gauge degrees of freedom and plays a role of connection with respect to additional Z^A directions. It is convenient to introduce anticommuting Z -differentials $dZ^A dZ^B = -dZ^B dZ^A$ to interpret $S_A(Z; Y; \mathcal{K} | x)$ as a Z -1-form,

$$S = dZ^A S_A(Z; Y; \mathcal{K} | x). \quad (6.3)$$

The variables $\mathcal{K} = (k, \bar{k})$ are Klein operators that satisfy (4.20) with $w^\alpha = (y^\alpha, z^\alpha, dz^\alpha)$, $\bar{w}^{\dot{\alpha}} = (\bar{y}^{\dot{\alpha}}, \bar{z}^{\dot{\alpha}}, d\bar{z}^{\dot{\alpha}})$.

The nonlinear HS equations are [2]

$$dW + W * W = 0, \quad (6.4)$$

$$dB + W * B - B * W = 0, \quad (6.5)$$

$$dS + W * S - S * W = 0, \quad (6.6)$$

$$S * B = B * S, \quad (6.7)$$

$$S * S = -i(dZ^A dZ_A + dz^\alpha dz_\alpha F_*(B)kv + d\bar{z}^{\dot{\alpha}} d\bar{z}_{\dot{\alpha}} \bar{F}_*(B)\bar{k}\bar{v}), \quad (6.8)$$

where $F_*(B)$ is some star-product function of the field B .

The simplest case of linear functions

$$F_*(B) = \eta B, \quad \bar{F}_*(B) = \bar{\eta} B, \quad (6.9)$$

where η is some phase factor (its absolute value can be absorbed into redefinition of B) leads to a class of pairwise nonequivalent nonlinear HS theories. The particular cases of $\eta = 1$ and $\eta = \exp \frac{i\pi}{2}$ are especially interesting, corresponding to so called A and B HS models. These two cases are distinguished by the property that they respect parity [22].

The associative star product $*$ acts on functions of two spinor variables

$$(f * g)(Z; Y) = \frac{1}{(2\pi)^4} \int d^4 U d^4 V \exp[iU^A V^B C_{AB}] f(Z + U; Y + U) g(Z - V; Y + V), \quad (6.10)$$

where $C_{AB} = (\epsilon_{\alpha\beta}, \bar{\epsilon}_{\dot{\alpha}\dot{\beta}})$ is the $4d$ charge conjugation matrix and U^A, V^B are real integration variables. It is normalized so that 1 is a unit element of the star-product algebra, *i.e.*, $f * 1 = 1 * f = f$. Star product (6.10) provides a particular realization of the Weyl algebra

$$[Y_A, Y_B]_* = -[Z_A, Z_B]_* = 2iC_{AB}, \quad [Y_A, Z_B]_* = 0 \quad (6.11)$$

($[a, b]_* = a * b - b * a$) resulting from the normal ordering with respect to the elements

$$b_A = \frac{1}{2i}(Y_A - Z_A), \quad a_A = \frac{1}{2}(Y_A + Z_A), \quad (6.12)$$

which satisfy

$$[a_A, a_B]_* = [b_A, b_B]_* = 0, \quad [a_A, b_B]_* = C_{AB} \quad (6.13)$$

and can be interpreted as creation and annihilation operators as is most evident from the relations

$$b_A * f(b, a) = b_A f(b, a), \quad f(b, a) * a_A = f(b, a) a_A. \quad (6.14)$$

From (6.10) it follows that functions $f(Y)$ form a proper subalgebra which is the centralizer of the elements Z_A . For Z -independent $f(Y)$ the star product (6.10) takes the form of the Weyl star product.

An important property of the star product (6.10) is that it admits the inner Klein operator

$$\Upsilon = \exp iZ_A Y^A, \quad (6.15)$$

which behaves as $(-1)^N$, where N is the spinor number operator. One can easily see that

$$\Upsilon * \Upsilon = 1, \quad (6.16)$$

$$\Upsilon * f(Z; Y) = f(-Z; -Y) * \Upsilon, \quad (6.17)$$

and

$$(\Upsilon * f)(Z; Y) = \exp iZ_A Y^A f(Y; Z). \quad (6.18)$$

The left and right inner Klein operators

$$v = \exp i z_\alpha y^\alpha, \quad \bar{v} = \exp i \bar{z}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}}, \quad (6.19)$$

which enter Eq. (6.8), act analogously on undotted and dotted spinors, respectively

$$(v * f)(z, \bar{z}; y, \bar{y}) = \exp i z_\alpha y^\alpha f(y, \bar{z}; z, \bar{y}), \quad (\bar{v} * f)(z, \bar{z}; y, \bar{y}) = \exp i \bar{z}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}} f(z, \bar{y}; y, \bar{z}), \quad (6.20)$$

$$v * f(z, \bar{z}; y, \bar{y}) = f(-z, \bar{z}; -y, \bar{y}) * v, \quad \bar{v} * f(z, \bar{z}; y, \bar{y}) = f(z, -\bar{z}; y, -\bar{y}) * \bar{v}, \quad (6.21)$$

$$v * v = \bar{v} * \bar{v} = 1, \quad v * \bar{v} = \bar{v} * v. \quad (6.22)$$

To analyze Eqs. (6.4)-(6.8) perturbatively, one has to linearize them around some vacuum solution. The simplest choice is

$$W_0(Z; Y|x) = W_0(Y|x), \quad S_0(Z; Y|x) = dZ^A Z_A, \quad B_0 = 0, \quad (6.23)$$

where $W_0(Y|x)$ is some solution of the flatness condition

$$dW_0(Y|x) + W_0(Y|x) * W_0(Y|x) = 0. \quad (6.24)$$

$W_0(Y|x)$ bilinear in Y^A describes AdS_4 . Linearization of the system (6.4)-(6.8) around this vacuum just reproduces the free field equations (4.7), (4.8) (for more detail see [2, 3]).

In the purely bosonic case where all fermion fields are zero, the operator $k\bar{k}$ remains central in the full nonlinear system. As a result, bosonic sector of the system (6.4)-(6.8) decomposes into two independent subsectors singled out by the projectors Π_\pm (4.21).

6.2 Spinor coordinates

An important feature of the system (6.4)-(6.8) is that Eqs. (6.4)-(6.6), that contain space-time differential d , are flatness conditions. As a result, the flows along space-time coordinates commute to Eqs. (6.7), (6.8). This has two consequences. One is that nontrivial dynamics is hidden entirely in the noncommutative twistor space of Z and Y [41]. Another is that the system remains consistent if original space-time coordinates $x^{\alpha\dot{\alpha}}$ are extended to a larger space. Of course, if the differential is extended as in (3.1) with the same vacuum connection, additional equations will simply mean that, up to gauge ambiguity, all fields are independent of the coordinates \hat{X} . However, the situation becomes more interesting if pullback of a vacuum connection to additional directions is nonzero.

As explained in Section 2.2.1, this can be achieved by introducing a connection with respect to some symmetry algebra \mathfrak{h} in the system. So far, vacuum connection was introduced for the AdS_4 algebra $sp(4|\mathbb{R})$. We believe that in the context of holographic interpretation of the AdS_4 HS theory it may be useful to extend $sp(4|\mathbb{R})$ to the Lie algebra with the generators

$$T_{AB} = -\frac{i}{2} Y_A Y_B, \quad t_A = Y_A \quad (6.25)$$

obeying commutation relations

$$[T_{AB}, T_{CD}] = C_{BC}T_{AD} + C_{AC}T_{BD} + C_{BD}T_{AC} + C_{AD}T_{BC}, \quad (6.26)$$

$$[T_{AB}, t_C] = C_{BC}t_A + C_{AC}t_B, \quad [t_A, t_B] = 2iC_{AB}. \quad (6.27)$$

(The central element on the r.h.s. of the second relation (6.27), which can be identified with \hbar in the Heisenberg algebra h_4 spanned by T_A , is set to unity.) Following [35], we call this Lie algebra $sph(4|\mathbb{R})$. It should not be confused with the superalgebra $osp(1,4)$ where Y_A are treated as supergenerators. Clearly, $sph(4|\mathbb{R}) = sp(4|\mathbb{R}) \oplus h_4$. Note that $sph(4|\mathbb{R})$ is a parabolic subalgebra of $sp(6|\mathbb{R})$.

This generalization is aimed at the extension to the action level of the construction of [25] explained in Section 5 where conserved currents were represented by closed forms in the correspondence space unifying space-time and spinor coordinates. As shown in [35], relevant geometry naturally results from the formulation of HS theory in an appropriate coset space of the group $SpH(4|\mathbb{R})$. The idea is to introduce additional commutative coordinates $u^{\underline{A}}$ associated with additional generators in $sph(4|\mathbb{R})$ compared to $sp(4|\mathbb{R})$. Namely, we set

$$X = (x^{\alpha\dot{\alpha}}, u^{\underline{A}}), \quad u^{\underline{A}} = (u^{\underline{\alpha}}, \bar{u}^{\dot{\underline{\alpha}}}). \quad (6.28)$$

Correspondingly, the space-time HS connection $W_x(Z; Y; \mathcal{K}|x) = dx^{\underline{n}}W_{\underline{n}}(Z; Y; \mathcal{K}|x)$ is extended to

$$W_X(Z; Y; \mathcal{K}|X) = W_x(Z; Y; \mathcal{K}|X) + W_u(Z; Y; \mathcal{K}|X), \quad (6.29)$$

$$W_x(Z; Y; \mathcal{K}|X) = dx^{\underline{n}}W_{\underline{n}}(Z; Y; \mathcal{K}|X), \quad W_u(Z; Y; \mathcal{K}|x) = du^{\underline{A}}W_{\underline{A}}(Z; Y; \mathcal{K}|X). \quad (6.30)$$

A vacuum connection can be chosen in the form

$$W_{0x}(Y|x) = \frac{i}{4}W_0^{AB}(x)Y_A Y_B, \quad W_{0u} = du^{\underline{A}}W_{0\underline{A}}^A(x)Y_A + i du^{\underline{A}}u^{\underline{B}}C_{\underline{AB}}, \quad (6.31)$$

where $W_{\underline{A}}^A(x)$ is a set of Killing spinors enumerated by the label \underline{A} , that satisfy the covariant constancy condition

$$dW_{0\underline{A}}^A(x) - W_{0\underline{A}}^A(x)W_{0\underline{B}}^B(x)C_{\underline{AB}} = 0. \quad (6.32)$$

As a consequence,

$$C_{\underline{AB}} = W_{0\underline{A}}^A(x)W_{0\underline{B}}^B(x)C_{AB} \quad (6.33)$$

is some constant antisymmetric matrix. Requiring

$$W_{0\underline{A}}^A(x_0) = \delta_{\underline{A}}^A \quad (6.34)$$

at some point x_0 , we achieve that $C_{\underline{AB}}(x) = C_{AB}$ for any x .

As explained in Section 13, a HS action should be described by some Q -closed 4-form where Q is the operator (2.7) associated with the unfolded form of HS equations resulting from the perturbative analysis of Eqs. (6.4)-(6.8). In other words, the action should be d -closed by virtue of these equations. Since Eqs. (6.4)-(6.8) are insensitive to particular detail of space-time, this should be true for any space-time coordinates that can be introduced within unfolded formulation. The idea of the spinor extension is to use coordinates $u^{\underline{A}}$ instead of $x^{\underline{n}}$ in the computations involving bulk-to-boundary propagators, which are expected to be simpler in the spinor space than in AdS_4 . A toy model for this mechanism is provided by the evaluation of conserved charges along the lines of [25].

7 AdS_4 HS theory as 3d conformal HS theory

As discussed in Section 3, unfolded formulation allows one to choose freely one or another space-time interpretation of the theory. To see HS AdS_4/CFT_3 holographic duality, consider pullback of all space-time differential forms (*i.e.*, curvatures and connections) to some 3d surface $\Sigma \in AdS_4$. This gives a subsystem of the original unfolded system in AdS_4 that now can be interpreted as a 3d system on Σ , being by construction equivalent to the original AdS_4 system. In the HS model of interest, 0-forms associated with AdS_4 massless fields acquire the meaning of 3d conserved currents. Indeed, from the 3d perspective, dotted and undotted indices carry equivalent Lorentz representation. Hence, the 3d pullback of Eq. (4.19) gives Eq. (5.15) for the generalized 3d conformal stress tensors. What is not guaranteed however is that conformal properties are manifest for dynamical variables inherited from the AdS_4 HS theory. Let us consider this point in some more detail.

For manifest conformal invariance, it is most convenient to introduce oscillators

$$y_\alpha^+ = \frac{1}{2}(y_\alpha - i\bar{y}_\alpha), \quad y_\alpha^- = \frac{1}{2}(\bar{y}_\alpha - iy_\alpha) \quad (7.1)$$

that satisfy

$$[y_\alpha^-, y^{+\beta}]_* = \delta_\alpha^\beta. \quad (7.2)$$

In the AdS_4 setup, the reality conditions imply $(y_\alpha^\pm)^\dagger = iy_\alpha^\pm$. In the conformal setup the appropriate reality conditions are

$$(y_\alpha^-)^\dagger = y^{+\alpha}. \quad (7.3)$$

Weyl star-product realization of the 3d conformal algebra $sp(4; \mathbb{R}) \sim o(3, 2)$ is

$$L^\alpha{}_\beta = y^{+\alpha}y_\beta^- - \frac{1}{2}\delta_\beta^\alpha y^{+\gamma}y_\gamma^-, \quad D = \frac{1}{2}y^{+\alpha}y_\alpha^-, \quad (7.4)$$

$$P_{\alpha\beta} = iy_\alpha^-y_\beta^-, \quad K^{\alpha\beta} = -iy^{+\alpha}y^{+\beta}. \quad (7.5)$$

Here generators of 3d Lorentz transformations $L^\alpha{}_\beta$ form $sp(2; \mathbb{R})$. D is the dilatation generator. $P_{\alpha\beta}$ and $K^{\alpha\beta}$ are generators of translations and special conformal transformations, respectively. By (7.2), conformal dimension of the HS gauge fields counts the difference of the numbers of pluses and minuses

$$[D, \omega(y^\pm|X)] = \frac{1}{2} \left(y^{+\alpha} \frac{\partial}{\partial y^{+\alpha}} - y_\alpha^- \frac{\partial}{\partial y_\alpha^-} \right) \omega(y^\pm|X). \quad (7.6)$$

3d conformal HS algebra coincides with the AdS_4 HS algebra rewritten in terms of oscillators y_α^\pm . In this form it was introduced in [52]. Hence, the pullback $\hat{\omega}(y^\pm|x)$ of the AdS_4 HS gauge fields $\omega(y^\pm|x)$ to Σ just gives the full set of 3d conformal HS gauge fields.

To make contact with the standard approach it is convenient to foliate AdS_4 so that

$$x^n = (\mathbf{x}^a, z), \quad (7.7)$$

where $\mathbf{x}^{\underline{a}}$ are coordinates of leafs ($\underline{a} = 0, 1, 2$) while z is a foliation parameter. Let $\hat{W}(y^\pm|\mathbf{x}, z) = d\mathbf{x}^{\underline{a}}\hat{W}_{\underline{a}}(y^\pm|\mathbf{x}, z)$ be pullback of $W(y^\pm|x)$ to a leaf at some z . At every z , the original AdS_4 HS theory gives rise to a $3d$ conformal HS theory with $3d$ conformal HS connections $\hat{W}(y^\pm|\mathbf{x}, z)$. Similarly, $\hat{W}_0(y^\pm|\mathbf{x}, z)$ inherited from some AdS_4 vacuum connection $W_0(y^\pm|x)$ provides a flat connection of the $3d$ conformal algebra $sp(4)$ on every leaf.

A less trivial part of the $3d$ reduction is due to the gluing terms in (4.18) and field equations (4.19) on the 0-forms C . First of all we observe that Eqs. (4.10), (4.15) give the AdS deformation of Eq. (5.6). Hence, in agreement with AdS/CFT correspondence prescription, 0-forms C describing massless fields in AdS_4 should be interpreted as conserved currents in the $3d$ conformal setup. However, this correspondence is not quite direct because the original 0-forms C do not transform properly under conformal transformations. Indeed, the dilatation generator D in the twisted adjoint representation is realized by anticommutator which gives a second-order differential operator

$$\{D, C\}_* = \left(y^{+\alpha} y_\alpha^- - \frac{1}{4} \frac{\partial^2}{\partial y^{+\alpha} \partial y_\alpha^-} \right) C \quad (7.8)$$

rather than the first-order operator (7.6) in the adjoint representation. This means that the set of fields C inherited from the AdS_4 theory is not manifestly conformal.

To solve this problem one should change variables from $C(y^\pm|x)$ to $T(y^\pm|x)$ to achieve that $T(y^\pm|x)$ transforms properly under dilatations. In fact, this issue is not specific to the conformal description, having its direct analogue on the AdS side where the fields $C(y^\pm|x)$ do not exhibit manifest decomposition in terms of eigenfunctions of the energy operator E which is holographically dual to D . As shown in [9], the transition from $C(y^\pm|x)$ to the basis which diagonalizes energy is nonlocal, forming a kind of non-unitary Bogolyubov transform. Similar transformation in the conformal setup is achieved by the transition from Weyl to Wick star product with respect to y^- and y^+ .

Let $f(y^\pm)$ be an element of the Weyl star-product algebra. The map from the Weyl star product

$$(f * g)(y^\pm) = \frac{1}{\pi^4} \int d^4 u^\pm d^4 v^\pm \exp 2(v_\alpha^- u^{+\alpha} - u_\alpha^- v^{+\alpha}) f(y^\pm + u^\pm) g(y^\pm + v^\pm) \quad (7.9)$$

to the Wick star product

$$(f_N \star g_N)(y^\pm) = \frac{1}{(2\pi)^2} \int d^4 u^\pm \exp(-u_\alpha^- u^{+\alpha}) f_N(y^+, y^- + u^-) g_N(y^+ + u^+, y^-) \quad (7.10)$$

is

$$f_N(y^\pm) = \frac{1}{\pi^2} \int d^4 u^\pm \exp(-2u_\alpha^- u^{+\alpha}) f(y^\pm + u^\pm) \quad (7.11)$$

or, equivalently,

$$f_N(y^\pm) = \exp \left(-\frac{1}{2} \epsilon^{\alpha\beta} \frac{\partial^2}{\partial y^{-\alpha} \partial y^{+\beta}} \right) f(y^\pm). \quad (7.12)$$

Wick star product has the properties

$$f_N(y^+) \star g_N(y^\pm) = f_N(y^+)g_N(y^\pm), \quad f_N(y^\pm) \star g_N(y^-) = f_N(y^\pm)g_N(y^-), \quad (7.13)$$

$$y_\alpha^- \star = y_\alpha^- + \frac{\partial}{\partial y^{+\alpha}}, \quad \star y_\alpha^+ = y_\alpha^+ - \frac{\overleftarrow{\partial}}{\partial y^{-\alpha}}. \quad (7.14)$$

Let us now apply these formulae to the dilatation operator D (7.4). First of all, we obtain that

$$D_N = \frac{1}{2}(y_\alpha^- y^{+\alpha} + 1). \quad (7.15)$$

Hence

$$D_N \star = \frac{1}{2} \left(y^{+\alpha} \frac{\partial}{\partial y^{+\alpha}} + y_\alpha^- y^{+\alpha} + 1 \right), \quad \star D_N = \frac{1}{2} \left(y_\alpha^- \frac{\partial}{\partial y_\alpha^-} + y_\alpha^- y^{+\alpha} + 1 \right). \quad (7.16)$$

In the twisted adjoint representation, the action of D_N therefore is

$$\{D_N, \dots\}_\star = \frac{1}{2} \left(y^{+\alpha} \frac{\partial}{\partial y^{+\alpha}} + y^{-\alpha} \frac{\partial}{\partial y^{-\alpha}} \right) + y_\alpha^- y^{+\alpha} + 1. \quad (7.17)$$

It remains to introduce

$$T(y^\pm|x) = \exp(-y_\alpha^- y^{+\alpha}) C_N(y^\pm|x) \quad (7.18)$$

to achieve that, in agreement with the interpretation of $T(y^\pm|x)$ as a $3d$ conformal current,

$$D_N(T(y^\pm)) = \frac{1}{2} \left(y^{+\alpha} \frac{\partial}{\partial y^{+\alpha}} + y^{-\alpha} \frac{\partial}{\partial y^{-\alpha}} + 2 \right) T(y^\pm). \quad (7.19)$$

Let us now look more closely at the action of the translation generator $P_{\alpha\beta}$ (7.5). To this end we observe that

$$k y_\alpha^\pm = \mp i y_\alpha^\mp k, \quad \bar{k} y_\alpha^\pm = \pm i y_\alpha^\mp \bar{k}. \quad (7.20)$$

Hence

$$P_{\alpha\beta} T(y^\pm) k = \frac{\partial^2}{\partial y^{+\alpha} \partial y^{+\beta}} T(y^\pm) k, \quad T(y^\pm) k P_{\alpha\beta} = -\frac{\partial^2}{\partial y^{-\alpha} \partial y^{-\beta}} T(y^\pm) k \quad (7.21)$$

and, using $3d$ Cartesian coordinates along with

$$\frac{\partial^2}{\partial y^{+\alpha} \partial y^{+\beta}} + \frac{\partial^2}{\partial y^{-\alpha} \partial y^{-\beta}} = 4 \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta}, \quad (7.22)$$

for the case where the pullback of the AdS_4 connection to Σ has $3d$ Cartesian form, the resulting equation on the $3d$ 0-forms acquires the form of rank-two equation (5.6)

$$D_{\mathbf{x}}^{tw} T(y, \bar{y}|x) = d_{\mathbf{x}} T(y, \bar{y}|x) + 4 d\mathbf{x}^{\alpha\beta} \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta} T(y, \bar{y}|x) = 0. \quad (7.23)$$

According to the analysis of σ_- cohomology of [21] summarized in Section 5, Eq. (7.23) describes two sets of conserved currents of all spins $s > 0$ and two spin zero branches distinguished by their symmetry under $y \leftrightarrow \bar{y}$. The symmetric branch is generated by $J^{sym}(x) = T(0|x)$ that has conformal dimension $\Delta = 1$. The antisymmetric branch is generated by $J^{asym}(x)$ (5.12). In accordance with (7.19), $\Delta(J^{asym}(x)) = 2$. These are just correct conformal dimensions for scalar currents associated with the two energy branches of the AdS_4 conformal scalar field.

In this setup, the AdS_4/CFT_3 HS duality takes place on every leaf of the z -foliation. However, boundary and bulk fields are related by the nonlocal transform from Weyl to Wick star product. Since unfolded equations relate space-time derivatives to those over y^\pm , nonlocal map in spinor variables is translated to the space-time nonlocality of the map of AdS_4 fields to conformal ones. Without using twistor variables it may be difficult to establish precise correspondence between the AdS_4 HS theory and its dual on any Σ . However, as shown in the next section, the holographic correspondence drastically simplifies if Σ is AdS_4 infinity, just reproducing the standard AdS/CFT recipe [5, 6] where fields at the boundary of AdS_{d+1} are identified with (sources for) currents in CFT_d .

Opposite $-+$ ordering choice leads to equivalent results with the exchange of y^+ and y^- . This means that D changes its sign while $P_{\alpha\beta}$ and $K^{\alpha\beta}$ should be reinterpreted as generators of special conformal transformations and translations, respectively. With these redefinitions consideration remains intact.

8 Holographic locality at infinity

8.1 Conformal foliation and Poincaré coordinates

Let M^d be a d -dimensional conformally flat space-time with local coordinates \mathbf{x} and $w_{\mathbf{x}}(\mathbf{x}) = w_{\mathbf{x}}^A T_A$ be some flat $o(d, 2)$ connection³

$$d_{\mathbf{x}} w_{\mathbf{x}}(\mathbf{x}) + w_{\mathbf{x}}(\mathbf{x}) w_{\mathbf{x}}(\mathbf{x}) = 0. \quad (8.1)$$

Let D be the dilatation generator among T_A which induces standard \mathbb{Z} grading on $o(d, 2)$ so that

$$[D, T_A] = \Delta(T_A) T_A, \quad (8.2)$$

where $\Delta(T_A)$ is conformal dimension of T_A which takes values ± 1 or 0 . Namely,

$$T_A = (L_{ab}, D, K_a, P_a), \quad (8.3)$$

where conformal dimensions of generators of Lorentz transformations L_{ab} , dilatations D , special conformal transformations K_a and translations P_a are

$$\Delta L = 0, \quad \Delta(D) = 0, \quad \Delta(K) = 1, \quad \Delta(P) = -1. \quad (8.4)$$

³That an $o(d, 2)$ connection $w_{\mathbf{x}}(\mathbf{x})$ is flat means that M^d endowed with the metric resulting from the vielbein associated with the P_a component of $w_{\mathbf{x}}(\mathbf{x})$ is (locally) conformally flat (see, e.g., [33]).

A particular flat connection which corresponds to Cartesian coordinates in M^d is

$$w_{\mathbf{x}}(\mathbf{x}) = d\mathbf{x}^a P_a . \quad (8.5)$$

Let us now introduce an additional coordinate z and differential dz so that $x = (\mathbf{x}, z)$ be the local coordinates of AdS_{d+1} . A conformally foliated connection $W(x)$ of AdS_{d+1} can be introduced as follows. The components of $W(x)$ with differentials $d\mathbf{x}$ are

$$W_{\mathbf{x}}^A(x) T_A = z^{\Delta(T_A)} w_{\mathbf{x}}^A(\mathbf{x}) T_A , \quad (8.6)$$

while the only nonzero dz component of the connection is associated with the dilatation generator D , having the form

$$W_z(x) = -z^{-1} dz D . \quad (8.7)$$

Clearly, so defined connection $W(x)$ is flat in (a local chart of) AdS_{d+1} . Poincaré coordinates result from this construction applied to the connection (8.5). It should be stressed, however, that this construction works for any $o(d, 2)$ flat connection $w_{\mathbf{x}}(\mathbf{x})$ in M^d . In particular, if $w_{\mathbf{x}}(\mathbf{x})$ itself is some AdS_d connection with nonzero components in $o(d-1, 2) \in o(d, 2)$, it can itself be represented in the form of conformal foliation with another foliation parameter z_1 , continuing the process of dimension reduction.

In spinor notations, local coordinates of AdS_4 are

$$x^{\alpha\dot{\alpha}} = (\mathbf{x}^{\alpha\dot{\alpha}}, -\frac{i}{2}\epsilon^{\alpha\dot{\alpha}} z^{-1}) , \quad (8.8)$$

where the symmetric part of $4d$ coordinates $\mathbf{x}^{\alpha\dot{\alpha}} = \mathbf{x}^{\dot{\alpha}\alpha}$ is identified with coordinates of Σ while z is the radial coordinate of AdS_4 . The appearance of $\epsilon^{\alpha\dot{\alpha}} = -\epsilon^{\dot{\alpha}\alpha}$ in the definition of z breaks $4d$ Lorentz symmetry $sp(2; \mathbb{C})$ to $3d$ Lorentz symmetry $sp(2; \mathbb{R})$ that acts on the both types of spinor indices.

Now we are in a position to analyze dynamical content of AdS_4 HS equations at $z \rightarrow 0$. We will work in terms of Weyl star product inherited from (6.10) in the sector of y^{\pm} variables. Using (8.5) and (7.5), (4.13), AdS_4 connection can be chosen in the form

$$W = \frac{i}{z} d\mathbf{x}^{\alpha\beta} y_{\alpha}^{-} y_{\beta}^{-} - \frac{dz}{2z} y_{\alpha}^{-} y^{+\alpha} , \quad (8.9)$$

which is equivalent to

$$W = \frac{1}{4z} (d\mathbf{x}^{\alpha\beta} (y_{\alpha} y_{\beta} - \bar{y}_{\alpha} \bar{y}_{\beta}) + 2id\mathbf{x}^{\alpha\beta} y_{\alpha} \bar{y}_{\beta} + dz y_{\alpha} \bar{y}^{\alpha}) . \quad (8.10)$$

By Eq. (4.13), AdS_4 vierbein and Lorentz connection are

$$e^{\alpha\dot{\alpha}} = \frac{1}{2z} dx^{\alpha\dot{\alpha}} , \quad \omega^{\alpha\beta} = -\frac{i}{4z} d\mathbf{x}^{\alpha\beta} , \quad \bar{\omega}^{\dot{\alpha}\dot{\beta}} = \frac{i}{4z} d\mathbf{x}^{\dot{\alpha}\dot{\beta}} . \quad (8.11)$$

8.2 0-forms

The unfolded equation on Weyl tensors in AdS_4 , which is the covariant constancy condition (4.19) in the twisted adjoint representation, decomposes into two equations with respect to the $3d$ coordinates $\mathbf{x}^{\alpha\beta}$ and z , respectively,

$$\left[d_{\mathbf{x}} + \frac{i}{z} d\mathbf{x}^{\alpha\beta} \left(y_{\alpha} \frac{\partial}{\partial y^{\beta}} - \bar{y}_{\alpha} \frac{\partial}{\partial \bar{y}^{\beta}} + y_{\alpha} \bar{y}_{\beta} - \frac{\partial^2}{\partial y^{\alpha} \partial \bar{y}^{\beta}} \right) \right] C(y, \bar{y} | \mathbf{x}, z) = 0, \quad (8.12)$$

$$\left[d_z + \frac{dz}{2z} \left(y_{\alpha} \bar{y}^{\alpha} - \epsilon^{\alpha\beta} \frac{\partial^2}{\partial y^{\alpha} \partial \bar{y}^{\beta}} \right) \right] C(y, \bar{y} | \mathbf{x}, z) = 0. \quad (8.13)$$

Consider Eq. (8.12) which should reproduce the rank-two equation (5.6) at $M = 2$. By the substitution

$$C(y, \bar{y} | \mathbf{x}, z) = \exp(y_{\alpha} \bar{y}^{\alpha}) \tilde{C}(y, \bar{y} | \mathbf{x}, z) \quad (8.14)$$

it amounts to

$$\left[d_{\mathbf{x}} - \frac{i}{z} d\mathbf{x}^{\alpha\beta} \frac{\partial^2}{\partial y^{\alpha} \partial \bar{y}^{\beta}} \right] \tilde{C}(y, \bar{y} | \mathbf{x}, z) = 0. \quad (8.15)$$

Rescaling the variables y^{α} and \bar{y}^{α} via the substitution

$$C(y, \bar{y} | \mathbf{x}, z) = z \exp(y_{\alpha} \bar{y}^{\alpha}) T(w, \bar{w} | \mathbf{x}, z), \quad (8.16)$$

where the overall factor of z is introduced for the future convenience and

$$w^{\alpha} = z^{1/2} y^{\alpha}, \quad \bar{w}^{\alpha} = z^{1/2} \bar{y}^{\alpha}, \quad (8.17)$$

we obtain that $T(w, \bar{w} | \mathbf{x}, z)$ satisfies the conformal-invariant rank-two unfolded equation

$$\left[d_{\mathbf{x}} - i d\mathbf{x}^{\alpha\beta} \frac{\partial^2}{\partial w^{\alpha} \partial \bar{w}^{\beta}} \right] T(w, \bar{w} | \mathbf{x}, z) = 0. \quad (8.18)$$

Substitution of (8.16) into Eq. (8.13) gives

$$\left(\frac{\partial}{\partial z} - \frac{1}{2} \epsilon^{\alpha\beta} \frac{\partial^2}{\partial w^{\alpha} \partial \bar{w}^{\beta}} \right) T(w, \bar{w} | \mathbf{x}, z) = 0. \quad (8.19)$$

Eqs. (8.18), (8.19) are linearized unfolded equations for 0-forms in AdS_4 HS theory in Poincaré coordinates. As anticipated, Eq. (8.18) describes $3d$ conserved currents. Eq. (8.19) tells us that contractions $w_{\alpha} \bar{w}^{\alpha}$ in $T(w, \bar{w} | \mathbf{x}, z)$ should carry appropriate powers of z . This conforms to the fact that, by virtue of the linearized equation (8.18), most of such components vanish as a consequence of the conservation equations for the currents J and \tilde{J} (5.9). Important exception is provided by J^{asym} (5.12) that describes the scalar mode of conformal dimension $\Delta = 2$ properly accounted by Eq. (8.19) (see also Section 8.4).

8.3 1-forms

Using background connection (8.9) and Weyl star product, we obtain in the sector of HS gauge fields

$$D_{\mathbf{x}}W(y^{\pm}|\mathbf{x}, z) = \left(d_{\mathbf{x}} + \frac{2i}{z} d\mathbf{x}^{\alpha\beta} y_{\alpha}^{-} \frac{\partial}{\partial y^{+\beta}} \right) W(y^{\pm}|\mathbf{x}, z), \quad (8.20)$$

$$D_z W(y^{\pm}|\mathbf{x}, z) = \left(d_z - \frac{dz}{2z} \left(y_{\alpha}^{+} \frac{\partial}{\partial y_{\alpha}^{+}} - y_{\alpha}^{-} \frac{\partial}{\partial y_{\alpha}^{-}} \right) \right) W(y^{\pm}|\mathbf{x}, z). \quad (8.21)$$

Setting

$$W(y^{\pm}|\mathbf{x}, z) = \Omega(v^{-}, w^{+}|\mathbf{x}, z), \quad (8.22)$$

where

$$v^{\pm} = z^{-1/2} y^{\pm}, \quad w^{\pm} = z^{1/2} y^{\pm}, \quad (8.23)$$

this gives

$$D_{\mathbf{x}}\Omega(v^{-}, w^{+}|\mathbf{x}, z) = \left(d_{\mathbf{x}} + 2id\mathbf{x}^{\alpha\beta} v_{\alpha}^{-} \frac{\partial}{\partial w^{+\beta}} \right) \Omega(v^{-}, w^{+}|\mathbf{x}, z), \quad (8.24)$$

$$D_z\Omega(v^{-}, w^{+}|\mathbf{x}, z) = d_z\Omega(v^{-}, w^{+}|\mathbf{x}, z). \quad (8.25)$$

Now consider Eq. (4.18) starting from $\mathbf{x}\mathbf{x}$ sector. Its r.h.s. takes the form

$$- \mathcal{H}^{\alpha\beta} \left(\eta \frac{\partial^2}{\partial \bar{w}^{\alpha} \partial \bar{w}^{\beta}} T^{1-i i}(0, \bar{w} | \mathbf{x}, z) + \bar{\eta} \frac{\partial^2}{\partial w^{\alpha} \partial w^{\beta}} T^{i 1-i}(w, 0 | \mathbf{x}, z) \right), \quad (8.26)$$

where

$$\mathcal{H}^{\alpha\beta} = \frac{1}{4} d\mathbf{x}^{\alpha}{}_{\gamma} \wedge d\mathbf{x}^{\beta\gamma}. \quad (8.27)$$

Let us stress that explicit dependence on z in Eq. (8.26) due to derivatives over y and \bar{y} , and the factors of z in the frame field (8.11) and the definition of T (8.16) cancel out. Using

$$w_{\alpha} = w_{\alpha}^{+} + izv_{\alpha}^{-}, \quad \bar{w}_{\alpha} = iw_{\alpha}^{+} + zv_{\alpha}^{-}, \quad (8.28)$$

Eq. (4.18) acquires the form

$$D_{\mathbf{x}}\Omega_{\mathbf{x}}^{jj}(v^{-}, w^{+}|\mathbf{x}, z) = \mathcal{H}^{\alpha\beta} \frac{\partial^2}{\partial w^{+\alpha} \partial w^{+\beta}} \left(\bar{\eta} T^{j 1-j}(w^{+} + izv^{-}, 0 | \mathbf{x}, z) - \eta T^{1-j j}(0, iw^{+} + zv^{-} | \mathbf{x}, z) \right). \quad (8.29)$$

In the limit $z \rightarrow 0$ this gives

$$D_{\mathbf{x}}\Omega_{\mathbf{x}}^{ii}(v^{-}, w^{+}|\mathbf{x}, 0) = \mathcal{H}^{\alpha\beta} \frac{\partial^2}{\partial w^{+\alpha} \partial w^{+\beta}} \mathcal{T}^{ii}(w^{+}, 0 | \mathbf{x}, 0), \quad (8.30)$$

where

$$\mathcal{T}^{jj}(w^{+}, w^{-} | \mathbf{x}, z) = \bar{\eta} T^{j 1-j}(w^{+}, w^{-} | \mathbf{x}, z) - \eta T^{1-j j}(-iw^{-}, iw^{+} | \mathbf{x}, z) \quad (8.31)$$

satisfies the rank-two equation (8.18)

$$\left[d_{\mathbf{x}} - id_{\mathbf{x}}^{\alpha\beta} \frac{\partial^2}{\partial w^{+\alpha} \partial w^{-\beta}} \right] \mathcal{T}^{ii}(w^+, w^- | \mathbf{x}, z) = 0. \quad (8.32)$$

Eqs (8.30), (8.32) are linearized unfolded equations of free $3d$ conformal HS theory that describes conserved currents \mathcal{T}^{ii} and conformal HS gauge fields $\Omega_{\mathbf{x}}^{ii}$. Being inherited from the nonlinear HS theory in AdS_4 , the full boundary theory should be a nonlinear conformal HS gauge theory of currents \mathcal{T}^{ii} interacting with gauge fields $\Omega_{\mathbf{x}}^{ii}$ of Chern-Simons type. We will come back to this issue in Section 9.

In the $z\mathbf{x}$ sector, Eq. (4.18) gives

$$\begin{aligned} & D_{\mathbf{x}} \Omega_z^{jj}(v^-, w^+ | \mathbf{x}, z) + D_z \Omega_{\mathbf{x}}^{jj}(v^-, w^+ | \mathbf{x}, z) = \\ & = -\frac{i}{2} d_{\mathbf{x}}^{\alpha\beta} dz \frac{\partial^2}{\partial w^{+\alpha} \partial w^{+\beta}} (\bar{\eta} T^{j1-j}(w^+ + izv^-, 0 | \mathbf{x}, z) + \eta T^{1-jj}(0, iw^+ + zv^- | \mathbf{x}, z)) \end{aligned} \quad (8.33)$$

Eqs. (8.19) and (8.33) determine z -evolution of $\Omega_{\mathbf{x}}^{jj}(v^-, w^+ | \mathbf{x}, z)$ and $\mathcal{T}^{jj}(w^{\pm} | \mathbf{x}, z)$. According to [53, 54, 55], supplemented with nonlinear corrections, these should be interpreted as renormalization group equations.

8.4 Weyl, Wick and Fock

Let us consider in more detail the relation between the Wick star-product formalism of Section 7 and the Weyl star-product formalism used in Section 8.

Naively, the boundary limit $z \rightarrow 0$ of the map (7.12) gives the identity operator since

$$f_N(w^{\pm} | z) = \exp \left(-2z\epsilon^{\alpha\beta} \frac{\partial^2}{\partial w^{-\alpha} \partial w^{+\beta}} \right) f(w^{\pm} | z). \quad (8.34)$$

This is however not true because of the exponential factors in (7.18) and (8.16) which are singular in the limit $z \rightarrow 0$ at w fixed. Moreover, formulas (7.18) and (8.16) seemingly do not match each other because the exponential factors look different taking into account that

$$y_{\alpha}^{-} y^{+\alpha} = -\frac{1}{2} y_{\alpha}^{-} \bar{y}^{\alpha}. \quad (8.35)$$

This is however just the effect of using different star products in the respective formulas.

Indeed, consider a Weyl star-product element of the form

$$c(y^{\pm}) = \exp(-2y_{\alpha}^{-} y^{+\alpha}) t(w^{\pm}) \quad (8.36)$$

which is an analogue of (8.16). Using (7.11), it is easy to see that

$$c_N(y^{\pm}) = \exp(-y_{\alpha}^{-} y^{+\alpha}) \frac{1}{\pi^2} \int d^4 u^{\pm} \exp(-u_{\alpha}^{-} u^{+\alpha}) t\left(\frac{1}{2}(w^{\pm} + z^{1/2} u^{\pm})\right). \quad (8.37)$$

Similarly to (8.34), integration over u^\pm trivializes at $z \rightarrow 0$ giving

$$c_N(y^\pm) \Big|_{z=0} = \exp(-y_\alpha^- y^{+\alpha}) t\left(\frac{1}{2}w^\pm\right). \quad (8.38)$$

The exponential factor in this formula just matches that in (7.18).

In fact, in the both of star products, the exponentials

$$F_N = \exp -y_\alpha^- y^{+\alpha}, \quad F = \exp -2y_\alpha^- y^{+\alpha} \quad (8.39)$$

provide star-product realization of the Fock vacuum that satisfies

$$y_\alpha^- * F = y_\alpha^- \star F_N = 0, \quad F * y_\alpha^+ = F_N \star y_\alpha^+ = 0. \quad (8.40)$$

Correspondingly, the substitution of the exponential as in (7.18) maps Wick star product \star to the operation \circ that describes action of normal-ordered operators in the Fock bimodule generated from the Fock vacuum (8.39)

$$f(y_\beta^-, y_\alpha^+) \circ T(y^\pm) = f\left(\frac{\partial}{\partial y^{+\beta}}, y_\alpha^+\right) T(y^\pm), \quad T(y^\pm) \circ f(y_\beta^-, y_\alpha^+) = f\left(y_\beta^-, -\frac{\partial}{\partial y^{-\alpha}}\right) T(y^\pm), \quad (8.41)$$

where derivatives $\frac{\partial}{\partial y^{\pm\beta}}$ act on $T(y^\pm)$.

Note that, in the conformal setup, the exponential factor in (8.16) trivializes for all primary fields except for J^{asym} (5.12) since all other primaries depend either only on y^- or only on y^+ . In the case of J^{asym} , the exponential factor accounts properly the asymptotic z -dependence of J^{asym} in accordance with its conformal dimension $\Delta = 2$.

Thus, the Wick and Weyl star-product descriptions match at AdS_4 infinity. Nonlocality of the holographic conformal map in the bulk trivializes at AdS_4 infinity, leading to the standard AdS/CFT prescription where boundary values of the HS gauge fields in the bulk are identified with sources to conformal operators of the boundary theory.

9 Towards nonlinear $3d$ conformal HS theory

Although unfolded formulation of the nonlinear $3d$ conformal HS theory is not yet known, it can be systematically reconstructed from the AdS_4 HS theory via extension of analysis of Section 8 to the nonlinear level. Detailed derivation of the nonlinear $3d$ conformal HS theory will be presented elsewhere. Here we only comment on its general structure.

Generally, the holographic image of the AdS_4 HS theory should be nonlinear. This is because AdS_4 HS gauge connections contain background and fluctuational parts as two pieces of the same field. The same should be true in its conformal version. To be formally consistent, a system of conformal HS equations should be nonlinear with the only exception for the case where r.h.s. of Eq. (8.30) can be zero in all orders. Let us explain this in some more detail.

To simplify notations, we consider the purely bosonic case with all fields even in spinor variables Y^A, Z^A, dZ^A , where one can discard the doubling of fields in the full nonlinear

system via truncation of the theory by the projectors (4.21). Correspondingly, in this section we discard the labels i, j writing $\Omega_{\mathbf{x}}$ and \mathcal{T} instead of $\Omega_{\mathbf{x}}^{ii}$ and \mathcal{T}^{ii} .

Let us decompose

$$\Omega_{\mathbf{x}}(v^-, w^+|\mathbf{x}) = \sum_{n,m=0}^{\infty} \Omega_{\mathbf{x}}^{n,m}(v^-, w^+|\mathbf{x}), \quad R_{1\mathbf{xx}} := D_{\mathbf{x}}\Omega_{\mathbf{x}} = \sum_{n,m=0}^{\infty} R_{1\mathbf{xx}}^{n,m}(v^-, w^+|\mathbf{x}), \quad (9.1)$$

$$A^{n,m}(v^-, w^+|\mathbf{x}) = A_{\alpha_1 \dots \alpha_n \beta_1 \dots \beta_m}^{n,m}(\mathbf{x}) v^{-\alpha_1} \dots v^{-\alpha_n} w^{+\beta_1} \dots w^{+\beta_m}. \quad (9.2)$$

Recall that a spin s gauge field is described by $\Omega_{\mathbf{x}}^{n,m}(v^-, w^+|\mathbf{x})$ with $n+m=2(s-1)$. A particularly important role is played by the fields

$$\Omega_{\mathbf{x}}^-(v^-|\mathbf{x}) = \sum_n \Omega_{\mathbf{x}}^{n,0}(v^-, 0|\mathbf{x}), \quad (9.3)$$

and curvatures

$$R_{\mathbf{xx}}^+(w^+|\mathbf{x}) = \sum_m R_{1\mathbf{xx}}^{0,m}(0, w^+|\mathbf{x}). \quad (9.4)$$

$\Omega_{\mathbf{x}}^-(v^-|\mathbf{x})$ and $R_{\mathbf{xx}}^+(w^+|\mathbf{x})$ belong to the subspaces of, respectively, lowest and highest vectors of the $3d$ conformal subalgebra $sp(4)$ of the $3d$ conformal HS algebra.

$\Omega_{\mathbf{x}}^-(v^-|\mathbf{x})$ is the generating function for dynamical conformal HS gauge fields as can be seen by virtue of the σ_- cohomology analysis with

$$\sigma_- = e^{\alpha\beta} P_{\alpha\beta} = 2id\mathbf{x}^{\alpha\beta} v_{\alpha}^- \frac{\partial}{\partial w^{+\beta}}. \quad (9.5)$$

Dynamical fields associated with $H^1(\sigma_-)$ are rank- s totally symmetric multispinor fields $\varphi_{\alpha_1 \dots \alpha_{2s}}(\mathbf{x})$ representing $\Omega_{\mathbf{x}}^-(v^-|\mathbf{x})$ up to Lorentz and dilatation HS gauge shift symmetries

$$\Omega_{\mathbf{x}}^-(v^-|\mathbf{x}) = e^{\alpha_1\beta_2} v^{-\alpha_3} \dots v^{-\alpha_{2s}} \varphi_{\alpha_1 \dots \alpha_{2s}}(\mathbf{x}). \quad (9.6)$$

$\varphi_{\alpha_1 \dots \alpha_{2s}}(\mathbf{x})$ provides $3d$ spinor realization of traceless conformal HS gauge fields introduced by Fradkin and Tseytlin for $d=4$ in [56]. Having the gauge transformation law

$$\delta\varphi_{\alpha_1 \dots \alpha_{2s}} = \partial_{(\alpha_1 \alpha_2} \varepsilon_{\alpha_3 \dots \alpha_{2s})}, \quad (9.7)$$

they are dual to conserved conformal currents, serving as sources for correlators of currents in the holographic interpretation.

At the linearized level, $R_{\mathbf{xx}}^+(w^+|\mathbf{x})$ is the part of the linearized conformal HS curvature that contains nontrivial gauge invariant combinations of derivatives of the dynamical fields $\varphi_{\alpha_1 \dots \alpha_{2s}}$. Namely, the σ_- cohomology analysis shows that the conditions

$$R_{1\mathbf{xx}}^{n,m} = 0, \quad n > 0 \quad (9.8)$$

give algebraic constraints which express fields $\Omega_{\mathbf{x}}^{n,m}$ via order- m derivatives of the dynamical fields, imposing no restrictions on the latter. Although most of components of $R_{\mathbf{xx}}^+(w^+|\mathbf{x})$

vanish by virtue of Bianchi identities applied to (9.8), some may remain nonzero. These are just those parametrized by the 0-forms $\mathcal{T}(w^+, 0 | \mathbf{x})$ on the r.h.s. of the unfolded equations (8.30) (discarding indices i). In fact, $\mathcal{T}(w^+, 0 | \mathbf{x})$ just represents $H^2(\sigma_-)$.

Important property of the unfolded system (8.30), (8.32) with currents $\mathcal{T}(w^+, w^- | \mathbf{x})$ treated as independent $3d$ fields is that it is off-shell. This means that the system (8.30), (8.32) expresses up to gauge transformations all fields $\Omega_{\mathbf{x}}(v^-, w^+ | \mathbf{x})$ and $\mathcal{T}(w^+, w^- | \mathbf{x})$ via derivatives of $\varphi_{\alpha_1 \dots \alpha_{2s}}$ imposing no restrictions on the latter. In particular, this means that Eqs. (8.32) are consequences of Eqs. (8.30) supplemented with constraints which express $\mathcal{T}(w^+, w^- | \mathbf{x})$ via derivatives of $\mathcal{T}(w^+, 0 | \mathbf{x})$. In this setup, the current conservation equation

$$\frac{\partial}{\partial \mathbf{x}^{\alpha\beta}} \frac{\partial^2}{\partial w_{\alpha}^+ \partial w_{\beta}^+} \mathcal{T}(w^+, 0 | \mathbf{x}) = 0 \quad (9.9)$$

follows from the expression for $\mathcal{T}(w^+, 0 | \mathbf{x})$ in terms of derivatives of $\varphi_{\alpha_1 \dots \alpha_{2s}}$ by virtue of Eq. (8.32). Other way around, given conserved current $\mathcal{T}(w^+, 0 | \mathbf{x})$, Eq. (8.32) determines dynamical fields $\varphi_{\alpha_1 \dots \alpha_{2s}}$ in terms of $\mathcal{T}(w^+, 0 | \mathbf{x})$ up to a gauge transformation.

That the $3d$ system (8.30), (8.32) is off-shell means that unrestricted $\varphi_{\alpha_1 \dots \alpha_{2s}}$ can be interpreted as arbitrary boundary values of the bulk HS gauge fields. It should be stressed that, being off-shell in $d = 3$, the system (8.30), (8.32) becomes on-shell in a larger space like AdS_4 or a space with additional spinor coordinates of Section 6.2. As discussed in Section 3, this is crucial for holographic interpretation of the theory.

It should be stressed that unfolded dynamics properly accounts asymptotic behavior of relativistic fields in AdS . We have seen this already for scalar field in AdS_4 where unfolded equations reproduces two asymptotic behaviors with $\Delta = 1$ and $\Delta = 2$. Analysis of this section extends this observation to any spin. Indeed, Eq. (7.19) shows that conserved currents have canonical conformal dimension $s + 1$ and hence asymptotic behavior z^{s+1} . From Eq. (8.30) it follows that $\Omega(0, w^+)$ has asymptotic behavior z^{s-1} , which is in agreement with (7.6), taking into account the fact (4.1) that the total number of spinor indices carried by a spin- s connection is $2(s - 1)$. From Eq. (8.23) it follows that $\Omega(v^-, 0)$ has asymptotic behavior z^{1-s} which is again in agreement with (7.6). Since the background frame field in (9.6) contains the factor of z^{-1} , $\varphi_{\alpha_1 \dots \alpha_{2s}}$ has asymptotic behavior z^{2-s} which is just the correct behavior of the boundary source.

Setting to zero $\mathcal{T}(w^+, 0 | \mathbf{x})$ imposes differential equations on the dynamical fields $\varphi_{\alpha_1 \dots \alpha_{2s}}$. Since Eq. (9.8) at $\mathcal{T}(w^+, 0 | \mathbf{x}) = 0$ is the linearized flatness condition, in this case the HS gauge fields become pure gauge. For nonzero $\mathcal{T}(w^+, 0 | \mathbf{x})$, conformal HS gauge fields are nontrivial. To see whether or not the theory can remain free beyond the linearized approximation one has to check whether or not the condition $R_{1\mathbf{x}\mathbf{x}}(v^-, w^+ | \mathbf{x}) = 0$ is consistent with the full nonlinear HS equations in AdS_4 . As discussed in more detail in Section 10, this indeed turns out to be possible for two particular truncations of HS models: one for A -model and another for B -model. Correspondingly, the related truncated HS theories turn out to be holographically dual to free boundary bosonic and fermionic theories in agreement with Klebanov-Polyakov [13] and Sezgin-Sundell [22] conjectures. This conclusion is also in agreement with Maldacena-Zhiboedov theorem [20] because in these cases $3d$ conformal HS

gauge fields decouple from the $3d$ currents. However, beyond these two cases, the boundary dual of AdS_4 HS theory is nonlinear.

One reason why the boundary theory should be nonlinear is that the conformal HS curvatures inherited from AdS_4 HS theory

$$R_{\mathbf{x}\mathbf{x}}(v^-, w^+ | \mathbf{x}) = d_{\mathbf{x}}\Omega_{\mathbf{x}}(v^-, w^+ | \mathbf{x}) + \Omega_{\mathbf{x}}(v^-, w^+ | \mathbf{x}) \star \Omega_{\mathbf{x}}(v^-, w^+ | \mathbf{x}) \quad (9.10)$$

are nonlinear. Here it is crucial that the rescalings (8.17) and (8.23) have opposite scalings in the radial coordinate z so that v^- and w^+ obey z -independent commutation relations

$$[v_{\alpha}^-, w^{+\beta}]_{\star} = \delta_{\alpha}^{\beta}. \quad (9.11)$$

Analogously, l.h.s. of Eq. (8.32) deforms to the covariant derivative in the twisted adjoint representation of the non-Abelian $3d$ conformal HS algebra. Since the substitution (7.18) for the current maps Wick star product to Fock product, nonlinear extension of Eq. (8.32) starts from the twisted adjoint covariant derivative

$$\tilde{D}T(w^{\pm}|x) = dT(w^{\pm}|x) + \Omega(\frac{\partial}{\partial w^{+\beta}}, w_{\alpha}^{+})T(w^{\pm}|x) - T(w^{\pm}|x)\Omega(-i\frac{\partial}{\partial w^{-\alpha}}, -iw^{-}|x), \quad (9.12)$$

where we used (7.20) and that $\Omega(-v^-, -w^+) = \Omega(v^-, w^+)$ for bosons.

It should be stressed that nonlinear terms in Eqs. (9.10) and (9.12) are z -independent, hence fully reproducing the non-Abelian structure of $3d$ conformal HS theory in the $z \rightarrow 0$ limit. The nonlinear deformation (9.12) of the twisted adjoint covariant derivative implies a nonlinear deformation of the rank-two unfolded equation (8.32) which, in turn, implies a nonlinear deformation of the conventional current conservation condition, hence not respecting conditions of Maldacena-Zhiboedov theorem [20].

The nonlinear deformation due to non-Abelian HS algebra is just a first step requiring further \mathcal{T} -dependent nonlinear deformation of Eqs. (8.30) and (8.32). Similarly to [27] one can search these corrections perturbatively in powers of \mathcal{T} . However, straightforward analysis is not simple. It seems more promising to try to guess a closed form of yet unknown nonlinear $3d$ HS conformal unfolded system as it was guessed for the AdS_4 HS system in [1, 2]. We plan to consider this problem elsewhere.

It may look a bit peculiar that the variables v^+ and v^- appear asymmetrically in Eqs. (8.30), (8.32). This happens because of choosing a particular y^+y^- Wick ordering in the star product (7.10) or, alternatively, a particular form of connection (8.9). Choosing the y^-y^+ ordering exchanges the roles of y^+ and y^- . Full nonlinear conformal HS theory is expected to describe both of these sectors on equal footing.

Another comment is that in the realization of conformal HS theory described so far the $3d$ conformal currents appear as independent dynamical objects whose properties are determined by the equations of the $3d$ conformal HS theory itself. This model does not capture $3d$ conformal scalar and spinor fields $\Phi^i(w^{\pm}|x)$ from which the currents $\mathcal{T}(w^{\pm}|x)$ can be built. We believe that this ingredient can also be incorporated into $3d$ conformal HS theory. Indeed, as shown in [57], $3d$ conformal scalar and spinor can be described as fields

$|\Phi^i(w^+|x)\rangle$ valued in the Fock module of the $3d$ conformal HS algebra. In these terms, free field equations for $3d$ conformal fields have the form

$$(d + \Omega_0(v^-, w^+|\mathbf{x})) \star \Phi^i(w^+|\mathbf{x}) \star F = 0. \quad (9.13)$$

An interesting problem for the future is to find a nonlinear $3d$ conformal HS theory for the full system of fields Ω , \mathcal{T} and Φ , which relates \mathcal{T} to proper bilinear combinations of Φ . Solution of this problem should clarify explicit relation of our construction to (generalized) boundary σ -model constructions of [13, 22].

10 Boundary conditions, reductions and AdS doubling

Standard AdS/CFT correspondence assumes certain boundary conditions at infinity. In terms of AdS_4 HS Weyl forms they relate $T^{j1-j}(w^+, 0 | \mathbf{x}, 0)$ and $T^{1-jj}(0, iw^+ | \mathbf{x}, 0)$. Let

$$\mathcal{A}^{jj}(w^+, w^- | \mathbf{x}) = T^{j1-j}(w^+, w^- | \mathbf{x}, 0) - T^{1-jj}(-iw^-, iw^+ | \mathbf{x}, 0), \quad (10.1)$$

$$\mathcal{B}^{jj}(w^+, w^- | \mathbf{x}) = T^{j1-j}(w^+, w^- | \mathbf{x}, 0) + T^{1-jj}(-iw^-, iw^+ | \mathbf{x}, 0). \quad (10.2)$$

Conditions

$$\mathcal{A}^{jj}(w^+, w^- | \mathbf{x}) = 0 \quad (10.3)$$

and

$$\mathcal{B}^{jj}(w^+, w^- | \mathbf{x}) = 0 \quad (10.4)$$

will be called \mathcal{A} and \mathcal{B} , respectively. We observe that that r.h.s. of Eq. (8.30) is zero for the \mathcal{A} boundary conditions in the A -model ($\eta = 1$) and for the \mathcal{B} conditions in the B -model ($\eta = i$). On the other hand, r.h.s. of Eq. (8.33), that determines z -evolution of the HS connection, is zero for the \mathcal{B} conditions in the A -model and for the \mathcal{A} conditions in the B -model. This suggests that the latter boundary conditions correspond to IR fixed points of the model.

\mathcal{A}^{jj} and \mathcal{B}^{jj} describe independent combinations of T^{j1-j} , *i.e.*, each of \mathcal{A} or \mathcal{B} conditions leaves some components of T^{j1-j} nonzero. For any other choice of relative coefficients on the r.h.s.s of Eqs. (10.1), (10.2), the corresponding conditions would be too strong, implying $T^{j1-j} = 0$.

The bosonic model where all fermions are zero decomposes into two independent systems projected out by the projectors Π_{\pm} (4.21). The currents, that contribute to the Π_+ model, are

$$\mathcal{A}_+(w^+, w^- | \mathbf{x}) = \sum_{j=0,1} \left(T^{j1-j}(w^+, w^- | \mathbf{x}, 0) - T^{1-jj}(-iw^-, iw^+ | \mathbf{x}, 0) \right), \quad (10.5)$$

$$\mathcal{B}_+(w^+, w^- | \mathbf{x}) = \sum_{j=0,1} \left(T^{j1-j}(w^+, w^- | \mathbf{x}, 0) + T^{1-jj}(-iw^-, iw^+ | \mathbf{x}, 0) \right). \quad (10.6)$$

In this case, \mathcal{A} condition implies that the current $J^{asym}(X)$ (5.12) is zero while $J^{sym}(0 | X)$ remains free. \mathcal{B} condition implies that $J^{sym}(0 | X) = 0$ and $J^{asym}(X)$ remains free. Hence \mathcal{A} and \mathcal{B} boundary conditions just distinguish between two scalar currents that have different conformal dimensions, namely, $\Delta(J^{sym}) = 1$ and $\Delta(J^{asym}) = 2$.

In the Π_- model, \mathcal{A} and \mathcal{B} conditions have opposite effect. Namely, \mathcal{A} condition implies that the current $J^{sym}(X)$ is zero and $J^{asym}(0 | X)$ remains free while \mathcal{B} condition implies that $J^{asym}(0 | X) = 0$ and $J^{sym}(X)$ remains free. This conclusion is in agreement with supersymmetry of the model in presence of fermions: for any type of boundary conditions, the supersymmetric model will contain both types of scalars, one in the Π_+ sector and another in the Π_- sector. \mathcal{A} and \mathcal{B} boundary conditions extend two different types of boundary conditions for scalar currents to currents of all spins.

If the r.h.s. of Eq. (8.30) is zero, it becomes the flatness condition for boundary HS gauge fields. In the gauge where nonzero boundary HS gauge fields belong to the conformal algebra $sp(4)$, the resulting theory describes free unfolded equations on boundary currents in some conformally flat background. In agreement with Klebanov-Polyakov [13] and Sezgin-Sundell [22] conjectures these two particular cases correspond to the free boundary models of conformal scalar or spinor in Π_+A and Π_-B or Π_+B and Π_-A -models, respectively.

In fact, the free boundary theories are dual to truncations of the full nonlinear HS theories in AdS_4 by the parity automorphism P that exchanges dotted and undotted spinors. Indeed, as observed in [22], the nonlinear HS equations (6.4)-(6.8) are P invariant provided that $P(B) = B$ in the A -model or $P(B) = -B$ in the B -model.

There is however an interesting and important subtlety in this consideration. Indeed, so defined P describes the reflection $z \rightarrow -z$ of the coordinate z as introduced in (8.8). Hence, to apply P , one Poincaré chart of AdS_4 has to be supplemented with another one with negative z to allow

$$P(z) = -z. \quad (10.7)$$

In fact, in our construction it is important that AdS_4 is doubled to contain two Poincaré charts related by P . Although, geometrically, P leaves AdS_4 invariant, the effect of this doubling is nontrivial because extension of solutions from one chart to another is not necessarily P -invariant. For example, this is the case in non P -invariant HS theories with $\eta^2 \neq \pm 1$. On the other hand, no boundary conditions at $z = 0$ should be imposed to define the action as integral over the doubled AdS space-time. In this setup holographic duality relates a bulk theory in the doubled AdS space to the “boundary theory” where all possible types of boundary fields $\phi_{bound}(\mathbf{x})$ contribute. In the unfolded dynamics approach, values of $\phi_{bound}(\mathbf{x})$ at $z = 0$ reconstruct all fields in the (doubled) bulk and hence values of the respective action functionals $S(\phi_{bound})$. Note that in this respect the situation with the surface $z = 0$ in the doubled bulk space is analogous to that with a regular $3d$ surface Σ inside bulk as discussed in Section 7.

We believe that the doubled bulk AdS/CFT setup, which follows naturally from unfolded dynamics, has general applicability for HS theories and beyond. The important issue of anomalies also fits naturally this problem setting as we discuss briefly in Section 13.

In terms of elementary oscillators of the AdS_4 HS theory, P acts as follows

$$P(y_\alpha) = \bar{y}_\alpha, \quad P(\bar{y}_\alpha) = -y_\alpha \quad (10.8)$$

which is equivalent to

$$P(y_\alpha^\pm) = \pm i y_\alpha^\pm. \quad (10.9)$$

Being defined in such a way that it maps y^+ and y^- to themselves, P is not involutive. Namely, $P^2 = F$, $F^2 = Id$, where F is the boson-fermion automorphism that changes a sign of fermions. Although, naively, this property obstructs consistent P -reduction of the HS theory in presence of fermions, this is not the case. Remarkably, this is just what doctor ordered to cure additional factors of i that appear due to the $z \rightarrow -z$ reflection of the factor of $z^{1/2}$ in the rescaling (8.17), so that (10.8) is replaced by

$$P(w_\alpha^\pm) = \pm w_\alpha^\pm. \quad (10.10)$$

So defined P admits two extensions P_\pm to the full nonlinear HS system

$$P_\pm(z_\alpha) = -\bar{z}_\alpha, \quad P_\pm(\bar{z}_\alpha) = z_\alpha, \quad P_\pm(dz^\alpha) = -\bar{d}\bar{z}^\alpha, \quad P_\pm(\bar{d}\bar{z}^\alpha) = dz^\alpha, \quad (10.11)$$

$$P_\pm(k) = \pm \bar{k}, \quad P_\pm(\bar{k}) = \pm k. \quad (10.12)$$

(Spinor coordinates z_α and \bar{z}_α should not be confused with the radial coordinate z of AdS_4 .) P_\pm leave invariant all nonlinear equations except for Eq. (6.8) which is not invariant for general η . However, Eq. (6.8) is invariant under P_+ and P_- , in the cases of A and B models, respectively. This allows us to truncate nonlinear A and B -models in AdS_4 by the conditions

$$P_\pm W = W, \quad P_\pm S = S, \quad P_\pm B = B. \quad (10.13)$$

Associated boundary theories are A and B -models with, respectively, \mathcal{A} and \mathcal{B} boundary conditions at the linearized level. Since in these cases boundary currents decouple from $3d$ conformal HS gauge fields at the linearized level and since the conditions (10.13) are consistent in all orders, the corresponding truncations of the bulk HS theory should correspond to the free boundary theories in all orders.

Other models and/or boundary conditions do not correspond to any consistent truncation of the full bulk theory. Even if the boundary conditions were imposed in such a way that the r.h.s. of Eq. (8.30) be zero in the lowest order, it will acquire non-zero higher-order corrections from the full nonlinear system. From the perspective of [58], these cases correspond to broken HS symmetry because the current conservation equations are deformed by nonlinear corrections, *i.e.*, the currents are not conserved in the conventional sense. From the bulk HS theory perspective this is the effect of a nonlinear deformation of the HS gauge transformation law rather than breaking of HS symmetry.

To summarize, except for the particular cases of \mathcal{A} boundary condition in the A -model and \mathcal{B} boundary condition in the B -model, all other possibilities correspond to nonlinear boundary conformal HS theories where boundary conformal HS gauge fields are sourced by

boundary currents. This leads to fully nonlinear boundary theories where currents interact via Chern-Simons type boundary conformal HS gauge fields. In particular, this happens for all HS theories with $\eta^2 \neq \pm 1$.

The holographic duality described in this paper works for any coupling constant in the HS theory, hence not referring to the $N \rightarrow \infty$ limit. In this respect it extends the Klebanov-Polyakov-Sezgin-Sundell conjecture on the critical $O(N)$ and Gross-Neveu models to finite N . Beyond the $N \rightarrow \infty$ limit, AdS_4 HS theory is shown to be dual to a nonlinear theory that describes HS interactions of boundary currents via 3d conformal HS gauge fields. It remains to be seen what is the relation of this boundary HS theory to critical $O(N)$ and Gross-Neveu models as well as to the models with arbitrary η discussed in [59, 60].

11 AdS_3/CFT_2

11.1 AdS_3 description

AdS_3/CFT_2 correspondence in HS theories has been extensively studied in [61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73]. It is interesting to reconsider this problem along the lines of the AdS_4/CFT_3 analysis of previous sections. For details of the nonlinear AdS_3 HS theory we refer the reader to [74] (see also [3]). Below we only need the linearized construction.

The AdS_3 algebra is semisimple: $o(2, 2) \sim sp(2; R) \oplus sp(2; R)$ with the diagonal subalgebra $sp(2; R) \sim o(2, 1)$ as Lorentz algebra. A particularly useful realization of AdS_3 generators is

$$L_{\alpha\beta} = \frac{1}{4i} \{\hat{y}_\alpha, \hat{y}_\beta\}, \quad P_{\alpha\beta} = \frac{1}{4i} \{\hat{y}_\alpha, \hat{y}_\beta\} \psi \quad (11.1)$$

with the generating elements \hat{y}_α and ψ obeying the relations $[\hat{y}_\alpha, \hat{y}_\beta] = 2i\epsilon_{\alpha\beta}$, $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$, $\epsilon_{12} = 1$ and

$$\psi^2 = 1, \quad [\psi, \hat{y}_\alpha] = 0. \quad (11.2)$$

$\Pi_\pm = \frac{1}{2}(1 \pm \psi)$ are projectors to the simple components of $o(2, 2)$.

In [75, 76, 77, 78, 79] it was shown that there exists a one-parametric class of infinite-dimensional algebras $hs(2; \nu)$ (ν is an arbitrary real parameter), all containing $sp(2)$ as a subalgebra. This allows one to define a class of HS algebras $g = hs(2; \nu) \oplus hs(2; \nu)$ which admit a useful realization in terms of deformed oscillators.

Consider associative algebra $Aq(2; \nu)$ [78] of elements of the form

$$f(\hat{y}, k) = \sum_{n=0}^{\infty} \sum_{A=0,1} \frac{1}{n!} f^{A\alpha_1 \dots \alpha_n}(k) \hat{y}_{\alpha_1} \dots \hat{y}_{\alpha_n} \quad (11.3)$$

under condition that the coefficients $f^{A\alpha_1 \dots \alpha_n}$ are symmetric with respect to the indices α_j and that the generating elements \hat{y}_α obey

$$[\hat{y}_\alpha, \hat{y}_\beta] = 2i\epsilon_{\alpha\beta}(1 + \nu k), \quad k\hat{y}_\alpha = -\hat{y}_\alpha k, \quad k^2 = 1, \quad (11.4)$$

where ν is an arbitrary constant (central element). In other words, $Aq(2; \nu)$ is the enveloping algebra of the relations (11.4) often called deformed oscillator algebra.

Its important property is that, for all ν ,

$$T_{\alpha\beta} = \frac{1}{4i} \{\hat{y}_\alpha, \hat{y}_\beta\} \quad (11.5)$$

obey $sp(2)$ commutation relations, rotating \hat{y}_α as a $sp(2)$ vector

$$[T_{\alpha\beta}, T_{\gamma\eta}] = \epsilon_{\alpha\gamma} T_{\beta\eta} + \epsilon_{\beta\gamma} T_{\alpha\eta} + \epsilon_{\alpha\eta} T_{\beta\gamma} + \epsilon_{\beta\eta} T_{\alpha\gamma}, \quad (11.6)$$

$$[T_{\alpha\beta}, \hat{y}_\gamma] = \epsilon_{\alpha\gamma} \hat{y}_\beta + \epsilon_{\beta\gamma} \hat{y}_\alpha. \quad (11.7)$$

Deformed oscillators were originally discovered by Wigner [80] who addressed the question whether it is possible to modify the commutation relations for usual oscillators a^\pm in such a way that the basic commutation relations $[H, a^\pm] = \pm a^\pm$, $H = \frac{1}{2}\{a^+, a^-\}$ remain intact. By analyzing this problem in the Fock-type space Wigner found a one-parametric deformation of the standard commutation relations which gives a particular representation of the commutation relations (11.4) with the identification $a^+ = \hat{y}_1$, $a^- = \frac{1}{2i}\hat{y}_2$, $H = T_{12}$ and $k = (-1)^N$ where N is the particle number operator. These commutation relations were discussed later on by many authors (see, e.g., [81, 82, 83]).

According to (11.5) and (11.7), the $sp(2)$ generated by $T_{\alpha\beta}$ extends to $osp(1, 2)$ via identification of supergenerators with \hat{y}_α . The quadratic Casimir operator of $osp(1, 2)$

$$C_2 = -\frac{1}{2}T_{\alpha\beta}T^{\alpha\beta} - \frac{i}{4}\hat{y}_\alpha\hat{y}^\alpha, \quad (11.8)$$

is

$$C_2 = -\frac{1}{4}(1 - \nu^2). \quad (11.9)$$

Thus, $Aq(2, \nu)$ is isomorphic to $U(osp(1, 2))/I_{C_2 + \frac{1}{4}(1 - \nu^2)}$, where the ideal $I_{C_2 + \mu}$ consists of elements proportional to $C_2 + \mu$, $\mu \in \mathbb{C}$. This has a number of important consequences. For example, any module of $osp(1, 2)$ with $C_2 = -\frac{1}{4}(1 - \nu^2)$ forms a module of $Aq(2, \nu)$ ($\nu \neq 0$) and vice versa. In particular this is the case for finite-dimensional modules corresponding to $\nu = 2l + 1$, $l \in \mathbf{Z}$ with $C_2 = l(l + 1)$.

The even subalgebra of $Aq(2; \nu)$ spanned by $f(\hat{y}, k)$ (11.3) obeying $f(\hat{y}, k) = f(-\hat{y}, k)$ decomposes into direct sum of two subalgebras $Aq_\pm^E(2; \nu)$ spanned by the elements $\Pi_\pm f(\hat{y}, k)$ with $f(-\hat{y}, k) = f(\hat{y}, k)$, $\Pi_\pm = \frac{1}{2}(1 \pm k)$. These algebras are isomorphic to $U(sp(2))/I_{C_2 + \frac{3 \pm 2\nu - \nu^2}{4}}$, where $C_2 = -\frac{1}{2}T_{\alpha\beta}T^{\alpha\beta}$ is the quadratic Casimir operator of $sp(2)$, and can be interpreted as (infinite-dimensional) algebras interpolating between ordinary finite-dimensional matrix algebras as discussed in [75, 79].

Algebra $o(2, 2) \sim sp(2) \oplus sp(2)$ can be spanned by ψ -dependent bilinears of the oscillators \hat{y} . Its HS extension results from allowing all powers of \hat{y} . HS gauge fields are

$$w(\hat{y}, \psi, k|x) = \sum_{A, B=0,1; n=0}^{\infty} \frac{1}{n!} w^{AB\alpha_1 \dots \alpha_n}(x) k^A \psi^B \hat{y}_{\alpha_1} \dots \hat{y}_{\alpha_n}. \quad (11.10)$$

$w^{AB\alpha_1\ldots\alpha_n}(x)$ describe 3d HS gauge fields of spin $\frac{1}{2}n$. HS curvatures have standard form

$$R(\hat{y}, \psi, k|x) = dw(\hat{y}, \psi, k|x) + w(\hat{y}, \psi, k|x)w(\hat{y}, \psi, k|x). \quad (11.11)$$

(This construction for ordinary (*i.e.*, $\nu=0$) oscillators was suggested in [84].) The labels $A = 0, 1$ and $B = 0, 1$ play different roles. A describes the doubling of all fields as a consequence of $N = 2$ supersymmetry in the theory. This doubling can be avoided in an appropriately truncated theory [74]. B distinguishes between the Lorentz-like ($B = 0$) and frame-like ($B = 1$) fields.

The 3d linearized system is simpler than the 4d one because, analogously to 3d gravity [86, 87], 3d HS fields do not propagate being of Chern-Simons type. Equivalent statement is that 3d HS fields admit no HS Weyl tensors. Consequently, the 3d Central On-Mass-Shell Theorem has the form

$$R_1(\hat{y}, \psi, k|x) := dw(\hat{y}, \psi, k|x) + w_0(\hat{y}, \psi, k|x)w(\hat{y}, \psi, k|x) + w(\hat{y}, \psi, k|x)w_0(\hat{y}, \psi, k|x) = 0, \quad (11.12)$$

$$\mathcal{D}_0 C(\hat{y}, \psi, k|x) := dC(\hat{y}, \psi, k|x) + w_0(\hat{y}, \psi, k|x)C(\hat{y}, \psi, k|x) - C(\hat{y}, \psi, k|x)w_0(\hat{y}, -\psi, k|x) = 0, \quad (11.13)$$

where $w_0(\hat{y}, \psi, k|x)$ is some AdS_3 flat connection.

As shown in [85], in the sector of 0-forms, (11.13) describes four massive scalars, $C(\hat{y}, \psi, k|x) = C(-\hat{y}, \psi, k|x)$, and four massive spinors, $C(\hat{y}, \psi, k|x) = -C(-\hat{y}, \psi, k|x)$, arranged into $N = 2$ 3d hypermultiplets. Masses M of matter fields are expressed in terms of λ and ν as follows [85]

$$M_{\pm}^2 = \lambda^2 \frac{\nu(\nu \mp 2)}{2} \quad (11.14)$$

for bosons, and

$$M_{\pm}^2 = \lambda^2 \frac{\nu^2}{2} \quad (11.15)$$

for fermions. Here $-\lambda^2$ is the cosmological constant of AdS_3 . The signs “ \pm ” refer to the projections $C^{\pm} = \Pi_{\pm} C$, $\Pi_{\pm} = \frac{1 \pm k}{2}$. Doubling of fields of the same mass is due to ψ ($\psi^2 = 1$) while that with mass splitting in the bosonic sector, is due to k . Component form of the covariant constancy conditions (11.13) was originally found in [85] (see also [3]).

11.2 CFT_2 description

Analysis of conformal version of AdS_3 HS theory is to some extent parallel to the AdS_4 case. Radial coordinate z is identified with $z = x^{12}$ while the boundary coordinates are $\mathbf{x} = x^{11}$ and $\tilde{\mathbf{x}} = x^{22}$

$$x^{\alpha\beta} = (\mathbf{x}^{\alpha\beta}, \sigma_1^{\alpha\beta} z), \quad \sigma_{1\alpha\beta} \mathbf{x}^{\alpha\beta} = 0, \quad \sigma_1^{12} = \sigma_1^{21} = 1. \quad (11.16)$$

The 3d conformal algebra $o(2, 2)$ as well as its HS extension decomposes into direct sum of two subalgebras $o(2, 2) = sp(2) \oplus \tilde{sp}(2)$

$$sp(2) : T_{\alpha\beta} = \frac{1}{8i}(1 + \psi)(\hat{y}_{\alpha}\hat{y}_{\beta} + \hat{y}_{\beta}\hat{y}_{\alpha}), \quad \tilde{sp}(2) : \tilde{T}_{\alpha\beta} = \frac{1}{8i}(1 - \psi)(\hat{y}_{\alpha}\hat{y}_{\beta} + \hat{y}_{\beta}\hat{y}_{\alpha}). \quad (11.17)$$

Lorentz and dilatation generators are defined by the relations

$$L - D = \frac{1}{2}\sigma_1^{\alpha\beta}T_{\alpha\beta}, \quad L + D = \frac{1}{2}\sigma_1^{\alpha\beta}\tilde{T}_{\alpha\beta}. \quad (11.18)$$

From here it follows that

$$[D, T_{22}] = -T_{22}, \quad [D, T_{11}] = T_{11}, \quad [D, T_{12}] = 0, \quad (11.19)$$

$$[D, \tilde{T}_{22}] = \tilde{T}_{22}, \quad [D, \tilde{T}_{11}] = -\tilde{T}_{11}, \quad [D, \tilde{T}_{12}] = 0. \quad (11.20)$$

In accordance with (8.4) we set

$$P = T_{22}, \quad \tilde{P} = \tilde{T}_{11}. \quad (11.21)$$

Poincaré foliated flat connection (8.6) is

$$W_0 = z^{-1}(d\mathbf{x}P + d\tilde{\mathbf{x}}\tilde{P} - dzD). \quad (11.22)$$

Manifest conformal invariance is achieved via transition from the Weyl star product in AdS_3 setup to the Fock bimodule realization with the Fock vacua \mathcal{F}_\pm that satisfy

$$y_1 \circ \mathcal{F}_\pm = \mathcal{F}_\pm \circ y_2 = 0, \quad k\mathcal{F}_\pm = \mathcal{F}_\pm k = \pm \mathcal{F}_\pm. \quad (11.23)$$

An element $F_\pm(y)$ of the Fock bimodule results from the vacuum \mathcal{F}_\pm via action of functions of y_2 from the left and functions of y_1 from the right. This gives

$$y_1 \circ F_\pm(y) = 2i\mathcal{D}_1 F_\pm(y), \quad y_2 \circ F_\pm(y) = y_2 F_\pm(y), \quad (11.24)$$

$$F_\pm(y) \circ y_1 = F_\pm(y)y_1, \quad F_\pm(y) \circ y_2 = 2iF_\pm(y)\overleftarrow{\mathcal{D}}_2, \quad (11.25)$$

where

$$\mathcal{D}_1 F_\pm(y_1, y_2) = \frac{\partial}{\partial y^1} F_\pm(y_1, y_2) \pm \frac{\nu}{2y_2} (F_\pm(y_1, y_2) - F_\pm(y_1, -y_2)), \quad (11.26)$$

$$F_\pm(y_1, y_2)\overleftarrow{\mathcal{D}}_2 = \frac{\partial}{\partial y^2} F_\pm(y_1, y_2) \pm \frac{\nu}{2y_1} (F_\pm(y_1, y_2) - F_\pm(-y_1, y_2)). \quad (11.27)$$

Note that \mathcal{D}_1 and $\overleftarrow{\mathcal{D}}_2$ are so-called Dunkl derivatives [88] of the two-body Calogero model.

In this setup, the system becomes manifestly conformal with homogeneous polynomials of y_α carrying definite conformal dimensions in the adjoint

$$[D, A_\pm(y)] = \frac{1}{2} \left(y^2 \frac{\partial}{\partial y^2} - y^1 \frac{\partial}{\partial y^1} \right) A_\pm(y), \quad [D, \tilde{A}_\pm(y)] = \frac{1}{2} \left(y^1 \frac{\partial}{\partial y^1} - y^2 \frac{\partial}{\partial y^2} \right) \tilde{A}_\pm(y) \quad (11.28)$$

and twisted adjoint representation

$$D(F_\pm(y)) = \frac{1}{2} \left(\left(y^\alpha \frac{\partial}{\partial y^\alpha} + 2(1 \pm \nu) \right) F_\pm(y) \mp \nu (F_\pm(-y_1, y_2) + F_\pm(y_1, -y_2)) \right), \quad (11.29)$$

$$D(\tilde{F}_\pm(y)) = -\frac{1}{2} \left(\left(y^\alpha \frac{\partial}{\partial y^\alpha} + 2(1 \pm \nu) \right) \tilde{F}_\pm(y) \mp \nu (\tilde{F}_\pm(-y_1, y_2) + \tilde{F}_\pm(y_1, -y_2)) \right). \quad (11.30)$$

Lorentz transformation has universal form in all cases

$$LA(y) = \frac{1}{2} \left(y^1 \frac{\partial}{\partial y^1} - y^2 \frac{\partial}{\partial y^2} \right) A(y). \quad (11.31)$$

Essential difference between AdS_3/CFT_2 and AdS_4/CFT_3 dualities is that in the latter case 0-forms C are glued to the HS curvatures at the linearized level by Eq. (4.18), that leads to the nontrivial gluing of $3d$ conformal HS currents to $3d$ conformal HS gauge fields via (2.27). In the AdS_3 HS gauge theory no gluing between HS gauge fields and 0-forms C occurs at the linear level. Moreover, no nontrivial gluing of this type is even possible in AdS_3 because $3d$ Chern-Simons HS gauge theory admits no Weyl tensor and its HS generalizations. This has a consequence that $2d$ conformal fields J associated with C do not source $2d$ conformal HS gauge fields at the linearized level which, in fact, allows conformal fields associated with C to have continuous conformal dimension parametrized by ν .

This does not however imply that the $2d$ conformal fields J and $2d$ conformal HS gauge fields are completely independent. AdS_3 HS gauge fields are sourced by the HS currents built from bilinears of the AdS_3 matter fields C which represent the $3d$ stress tensor and its HS generalizations. From the CFT_2 dual viewpoint this means that $2d$ conformal HS curvatures will receive sources $T \sim JJ$ starting from the second order in J where T is a HS generalization of the stress tensor built from the currents J .

To obtain z -independent $2d$ equations from the AdS_3 HS theory one should again properly rescale the oscillators similarly to (8.17) and (8.23). To simplify formulae we abuse notations denoting the rescaled variables by y . Then $2d$ HS field equations have the following structure in the lowest order in which 0-forms contribute to r.h.s of the equations for HS gauge fields

$$R(y, k, \psi|\mathbf{x}, \tilde{\mathbf{x}}) = d\mathbf{x}d\tilde{\mathbf{x}} \frac{\partial^2}{\partial y^1 \partial y^2} (T(y_1, y_2, k, \psi|\mathbf{x}) + \tilde{T}(y_1, y_2, k, \psi|\tilde{\mathbf{x}})) + \dots, \quad (11.32)$$

where the HS curvature is

$$R(y, k, \psi|\mathbf{x}, \tilde{\mathbf{x}}) = d\Omega(y, k, \psi|\mathbf{x}, \tilde{\mathbf{x}}) + \Omega(y, k, \psi|\mathbf{x}, \tilde{\mathbf{x}}) \star \Omega(y, k, \psi|\mathbf{x}, \tilde{\mathbf{x}}) \quad (11.33)$$

with \star denoting the non-commutative product of deformed oscillators. Equation (11.32) is to some extent analogous to Eq. (8.33) if T would be treated as an independent field. More precisely, it is analogous to the $4d$ equations found in [89] where current interactions of $4d$ massless fields of all spins were constructed.

It remains to see to which extent the scheme sketched above reproduces the AdS_3/CFT_2 HS duality conjectures of [63, 67]. The construction of conformal currents T in terms of bilinears of the $2d$ fields J resulting from the $3d$ fields C is analogous to Sugawara construction considered in [67]. Correspondingly, the operator product of conformal conserved currents T is expected to reproduce the W_λ algebra [90, 64] with $\lambda = \frac{1 \pm \nu}{2}$. Note that at the classical level the construction of the nonlinear ν -dependent W algebra in terms of HS algebras, which is anticipated to be equivalent to the W_λ algebras of [90, 64], was found in [91]. In any case, a conformal theory dual to the AdS_3 HS theory should be nonlinear.

12 Higher-spin theory and quantum mechanics

Unfolded dynamics provides a powerful direct tool elucidating duality between theories in various dimensions, sometimes going beyond the conventional framework of AdS_{d+1}/CFT_d duality [4, 5, 6]. For instance, one can consider a chain of AdS_{n+1}/AdS_n dualities as conjectured in [9] (see also interesting recent work [92]), using the chain of Poincaré foliations (8.6), or, alternatively, by going directly from a higher dimension to the lower one. An intriguing example of the latter option considered in this section is provided by the duality between HS theories in the matrix space \mathcal{M}_M , formulated originally in the unfolded form in [9], and non-relativistic quantum mechanics. This consideration is closely related to the recent analysis of symmetries of quantum mechanical models in [93, 94, 95].

Via appropriate rescaling and complexification of variables, the rank-one equation (5.1) in \mathcal{M}_M can be rewritten in the form

$$\left(i\hbar\frac{\partial}{\partial X^{AB}} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial Y^A\partial Y^B}\right)\Psi(Y|X) = 0, \quad A, B = 1, \dots, M. \quad (12.1)$$

(Note that the factor of i in this equation naturally appears in the analysis of HS equations in Siegel space [25]). As discussed in Section 2.3, maximal symmetries of the free unfolded equations coincide with algebra $l^{max}(V)$ (of commutators) of endomorphisms of the space V where the 0-forms $\Psi(Y|X)$ at any $X = X_0$ are valued. Hence, symmetries of the equations (12.1) are generated by various operators in the space F of functions of Y^A .

Generally, to specify a space of operators in the functional space one has to specify their properties in some more detail. To respect relativistic symmetries, we should consider the space of differential operators with polynomial coefficients in Y^A which is equivalent to the Fock space realization with the “oscillators” P_A and Y^B that satisfy

$$[P_A, Y^B] = \delta_A^B, \quad [P_A, P_B] = 0, \quad [Y^A, Y^B] = 0. \quad (12.2)$$

In this terms, F is the space of vectors $f(Y)|0\rangle$ induced from the vacuum $|0\rangle$ satisfying

$$P_A|0\rangle = 0. \quad (12.3)$$

Hence, the symmetry algebra of Eq. (12.1) is generated by various polynomials of P_A and Y^B . This is the generalized conformal HS algebra considered in [9]. It contains $sp(2M)$ generated by

$$K^{AB} = Y^A Y^B, \quad L^A{}_B = \{Y^A, P_B\}, \quad P_{AB} = P_A P_B. \quad (12.4)$$

The X -dependence of global HS transformations determined by Eq. (2.21) was found in [9].

In [42] it was shown that time-like directions in \mathcal{M}_M are associated with positive-definite X^{AB} . In particular one can set

$$X^{AB} = tM\delta^{AB}, \quad (12.5)$$

where t is the time evolution parameter. Restriction of Eq. (12.1) to t gives usual M -dimensional Schrodinger equation

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\delta^{AB}\frac{\partial^2}{\partial Y^A\partial Y^B}\right)\Psi(Y|t) = 0, \quad (12.6)$$

where Y^A are now interpreted as coordinates of the Galilean space.

From general properties of unfolded formulation discussed in Section 2 it follows that relativistic rank-one equations in \mathcal{M}_M are equivalent to the nonrelativistic Schrodinger equation in M dimension. The cases of $M = 2$ and $M = 4$ are particularly interesting from the relativistic field theory perspective. Eq. (5.1) with $M = 2$ describes massless scalar ($\Psi(Y|X) = \Psi(-Y|X)$) and spinor ($\Psi(Y|X) = -\Psi(-Y|X)$) in 2+1 dimension. Eq. (5.1) with $M = 4$ describes massless particles of all integer ($\Psi(Y|X) = \Psi(-Y|X)$) and half-integer ($\Psi(Y|X) = -\Psi(-Y|X)$) spins in 3+1 dimension [9].

It should be noted that relativistic systems in \mathcal{M}_M are conformal [42, 96]. In particular, $sp(4)$ is just the $3d$ conformal algebra while $sp(8)$ contains the $4d$ conformal algebra $su(2, 2)$ as a subalgebra. This immediately implies that these algebras do act on solutions of the respective non-relativistic field equations as well as the full Weyl algebra of operators built from P_A and Y^A . However this action does not look geometric in terms of twistor variables Y^A interpreted as space coordinates of nonrelativistic quantum mechanics. (More precisely, beyond free field level, these are coordinates u^A introduced in Section 6.2.) Other way around, nonrelativistic symmetries, which act geometrically in terms of nonrelativistic coordinates Y^A , look nongeometric in terms of relativistic coordinates.

This is manifestation of a very general situation. In the unfolded dynamics approach it is easy to introduce coordinates in which any symmetry h of a given system acts geometrically by introducing an appropriate non-zero flat connection of h . However different symmetries require different coordinates (spaces) and connections. Description of the same system in different space-times gives holographically dual theories. Being obvious in unfolded dynamics approach, where it refers to the same twistor space (which is the space of Y^A in the quantum-mechanical model of interest), in other approaches holographic duality may look obscure.

Eq. (12.6) is Schrodinger equation for free nonrelativistic particle. One may wonder what if the system is deformed by a potential? In the framework of unfolded dynamics, this does not affect the consideration much, at least formally. Indeed, in presence of potential $U(Y)$, the equation

$$\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \delta^{AB} \frac{\partial^2}{\partial Y^A \partial Y^B} - U(Y) \right) \Psi(Y|t) = 0 \quad (12.7)$$

remains linear, hence exhibiting infinite symmetries. In the spirit of unfolded dynamics, it can be interpreted as flatness condition

$$D\Psi(Y|t) = 0, \quad D = dt \frac{\partial}{\partial t} + \Omega, \quad \Omega = i\hbar^{-1} dt H, \quad H = -\frac{\hbar^2}{2m} \delta^{AB} \frac{\partial^2}{\partial Y^A \partial Y^B} + U(Y). \quad (12.8)$$

In the one-dimensional case with the single coordinate t , any connection is flat, *i.e.*, the compatibility conditions for Eq. (12.8) are trivially satisfied. Hence it can be represented in the pure gauge form which is simply

$$\Omega = \exp(-i\hbar^{-1} H t) d \exp(i\hbar^{-1} H t). \quad (12.9)$$

The same similarity transform relates symmetries of the $H = 0$ system to those of $H \neq 0$.

Other way around, any HS connection $\Omega(Y|X)$ in \mathcal{M}_M (not necessarily flat) generates a flat connection Ω_t as its pullback to the time arrow. Hence any HS geometry is holographically dual to some quantum mechanics. For example, from Eq. (4.15) we observe that an appropriate λ -dependent rescaling maps *AdS* geometry to the harmonic potential

$$U(Y) = \frac{1}{2}m\omega^2 Y^A Y^B \delta_{AB}, \quad (12.10)$$

where the coupling constant is proportional the cosmological constant $-\Lambda \sim \lambda^2$

$$\frac{1}{2}m\omega^2 = \lambda^2. \quad (12.11)$$

On the other hand, *dS* geometry is holographically dual to the inverted harmonic potential with negative ω^2 , that is of course not too surprising in the context of inflation.

The correspondence between relativistic systems in higher dimensions and quantum mechanics is not just formal. In particular, these holographically dual systems have the same spectra. Namely, by virtue of unfolded equations, the spectrum of states of free relativistic massless particles of all spins in 3+1 dimension is identical to that of four quantum harmonic oscillators, while the spectrum of massless particles in 2+1 dimension is the same as of two harmonic oscillators. Finite-dimensional Schrodinger algebra of nonrelativistic symmetries of Schrodinger equation (see e.g. [95] and references therein) form a subalgebra of the algebra $sph(4|\mathbb{R})$ of Section 6.2. In particular, so-called mass operator \hat{M} is represented by the central element of the Heisenberg subalgebra of $sph(4|\mathbb{R})$.

Let us note that the duality of relativistic and nonrelativistic equations allows a natural interpretation for such a standard tool for the study of relativistic equations as oscillator realization of relativistic symmetries extensively used for the group-theoretic analysis of relativistic theories [97, 98]. From the holographic point of view pursued in this paper, it results from the dual realization of the relativistic system in terms of its nonrelativistic cousin. Moreover, not only symmetries of holographically dual relativistic and non-relativistic systems are the same. Their conserved currents coincide as well. This is a simple consequence of the analysis of [25] summarized in Section 5. The key fact is that a differential $2M$ -form Ω (5.16) is closed in the correspondence space unifying (relativistic) space-time \mathcal{M}_M with coordinates X^{AB} with the twistor space \mathbf{T} (non-relativistic space-time) with coordinates Y^A . As a result, a conserved charge can be evaluated both in \mathcal{M}_M and in \mathbf{T} . In the first case, it gives a relativistic conserved charge in \mathcal{M}_M while in the second case it appears as a nonrelativistic conserved charge in \mathbf{T} . In fact, higher conserved charges of nonrelativistic quantum mechanics constructed recently in [95] just coincide with those resulting from the pullback of Ω to \mathbf{T} .

Surprisingly enough, equivalence of relativistic and non-relativistic systems described above acquires interpretation of a Penrose transform induced by the unfolded equations. This should have much in common with the interpretation of non-relativistic physics as the relativistic one in the light-cone higher-dimensional system (see [95] and references therein).

13 Towards off-shell formulation

The property underlying holographic duality is that dynamics of universal unfolded systems is characterized entirely by the differential Q (2.7) defined on the “target space” of dynamical variables independently of the original space-time. In particular, invariants like actions and conserved charges are characterized by Q -cohomology [26].

First suppose that the system (2.2) is off-shell. As shown in [26], a gauge invariant action is an integral over a d -cycle M^d

$$S = \int_{M^d} \mathcal{L}(W). \quad (13.1)$$

of some Q -closed d -form *Lagrangian function* $\mathcal{L}(W)$

$$Q\mathcal{L} = 0 : \quad G^\alpha(W) \frac{\partial}{\partial W^\alpha} \mathcal{L}(W) = 0. \quad (13.2)$$

It is easy to see that, being Q -closed, S is invariant under the gauge transformations (2.9). If \mathcal{L} is Q -exact, by virtue of (2.8) it is d -exact, *i.e.* nontrivial invariant actions represent Q cohomology of the system in question.

If the system is on-shell and \mathcal{L} represents $H^p(Q)$, the same formula describes a conserved charge as an integral over a p -cycle Σ

$$q = \int_{\Sigma} \mathcal{L}(W). \quad (13.3)$$

Examples of application of this construction were given in [26]. Let us stress that, the analysis in terms of Q -cohomology applies to both linear and nonlinear unfolded systems.

In unfolded dynamics, Noether current interactions are directly related to conserved currents. In the case of interest they result from the expression for the conserved charge (5.17), (5.19). For example, in the case of $M = 2$, since the 4-form Ω (5.17) is closed for any \tilde{T}_η (5.19) with $\eta(W, Y|x)$ that satisfies Eq. (5.15), the 5-form

$$L = \Omega(W, Y|X) dW_\beta dW^\beta \left(i W_\beta dX^{\alpha\beta} - dY^\alpha \right) (i W^\gamma dX^\gamma_\alpha - dY_\alpha) \\ \int dU_\gamma dU^\gamma \exp -i W_\alpha U^\alpha J(U, Y|X) \quad (13.4)$$

is closed up to J^2 terms by virtue of the unfolded equations (8.30).

Let us look more closely at the relation between on-shell and off-shell systems. Let W_{on}^Ω be a set of forms of some on-shell system. Its off-shell extension should contain additional fields E^a that appear on the r.h.s.s of the field equations to replace the differential equations by constraints expressing new fields via l.h.s.s of the field equations. Abusing notation we can write

$$L^i(W(x)) = \mathcal{E}^i(x), \quad (13.5)$$

where $L^i(W(x))$ describes l.h.s.s of the dynamical equations on W^Ω . Note that \mathcal{E}^i is a part of the full set of E^a since, in unfolded dynamics, E^a contains \mathcal{E}^i along with all their derivatives.

We will call \mathcal{E}^i *primary off-shell fields*, saying that they glue the field equations of the on-shell system in question. This is equivalent to the statement that primary off-shell fields match the on-shell σ_- cohomology associated with the l.h.s of the field equations to enforce the corresponding cohomology of the off-shell system be zero.

The off-shell system with $W_{off}^A = (W_{on}^\Omega, E^a)$ is such that $E^a = 0$ puts the off-shell system on shell. This means that the off-shell system is described by such $G^A(W_{off})$ that both

$$Q^{off} = G^A(W_{off}) \frac{\partial}{\partial W_{off}^A} = G^\alpha(W_{on}, E) \frac{\partial}{\partial W_{on}^\alpha} + G^a(W_{on}, E) \frac{\partial}{\partial E^a} \quad (13.6)$$

and

$$Q^{on} = G^\alpha(W_{on}, 0) \frac{\partial}{\partial W_{on}^\alpha} \quad (13.7)$$

are nilpotent

$$Q^{off} Q^{off} = Q^{on} Q^{on} = 0. \quad (13.8)$$

This is a consequence of the property

$$G^a(W_{on}, 0) = 0, \quad (13.9)$$

which should be true for any off-shell extension. Indeed, otherwise, the on-shell fields W_{on}^α would source the fields E^a not allowing to put the system on shell. Field equations imply $\mathcal{E}^i = 0$ and hence

$$E^a = 0. \quad (13.10)$$

To extend the on-shell analysis of this paper to the full quantum level an off-shell extension of the system has to be considered. This problem has not been yet solved in a fully satisfactory way. An interesting hint from the analysis of [89] is that the system, that describes current interactions of $4d$ massless fields, can be viewed as the $4d$ off-shell system with the current fields J interpreted as off-shell fields $\mathcal{E}^i(x)$. On the other hand, the same fields can be interpreted either as describing two-particle states in the system or as free $6d$ fields. This suggests the idea that proper account of off-shell quantum effects in terms of unfolded dynamics may result from consideration of the theory in higher and higher dimensions, allowing to interpret quantum-mechanical effects as classical dynamics in an infinite-dimensional space that has enough room to describe all multiparticle states of the system.

In a more traditional fashion, if an off-shell action is available in the unfolded formulation, it can be used to produce generating functionals in the standard path integral approach. Again, the idea is that using that an action functional is closed in an appropriate correspondence space which extends space-time with some twistor coordinates, the integration can be performed in the twistor space for all holographically dual theories. In that case, various holographic interpretations of the same generating functional will be fully equivalent.

Actions of the form (13.1) are also appropriate for the analysis of anomalies in the formulation in the doubled bulk space suggested in Section 10. Although naive interpretation of the action (13.1) may be ill-defined because of divergencies at $z = 0$ it can be regularized via deformation of the integration contour to the complex plane in z , say, via substitution $z \rightarrow$

$z + i\epsilon$. Anomalous terms will be associated with singularities in ϵ . In fact, complexification of matrix coordinates X^{AB} in HS theories has been used in [25] to regularize integrals for HS conserved charges analogous to the action integral (13.1) where X^{AB} were complexified to \mathcal{Z}^{AB} from the upper Siegel half-space. In the example of [25] it was shown that the charge integrals are independent of variations of a complex integration contour away from singularity. If the same happens in HS theories this would imply that the regularized action is independent of ϵ hence being anomaly free (though it may be dependent on the contour homotopy class). The same time, independence of local contour variation implies Q -closure of the action (13.1) which, in turn, implies its gauge invariance [26]. It would be interesting to see how this scheme works in off-shell HS theories.

Note that for HS theories formulated in matrix space $M_{\mathcal{M}}$, regularization via deformation to Siegel space has deep meaning in various respects [25]. In particular, solutions analytic in upper and lower Siegel spaces correspond, respectively, to particles and antiparticles. Also solutions of HS equations from the upper Siegel half-space, that are periodic in the Y^A , are closely related to Riemann theta-functions where complexified coordinates \mathcal{Z}^{AB} acquire the meaning of a period matrix.

14 Conclusion

In this paper it is demonstrated how holographic duality results from different interpretations of one and the same theory. This phenomenon is very general and applies to any theory (not necessarily conformal). To establish holographic duality it is most useful to reformulate a theory in the unfolded form [27] of coordinate independent first-order equations formulated in terms of exterior differential as space-time derivative and differential forms as field variables. Once such a formulation is achieved, one can play freely with space-time dimension, adding or removing coordinates without changing dynamical content of the theory. This provides a vast variety of differently looking models in space-times of different dimensions which however are by construction locally equivalent. Since unfolding machinery applies to any theory, every model belongs to a class of holographically equivalent models.

Since HS theories were originally formulated within the unfolded dynamics approach, they provide a natural arena illustrating this phenomenon. In this paper we focused on the AdS_4/CFT_3 and AdS_3/CFT_2 HS dualities. The latter was put forward in [61, 62, 63, 67]. The former was conjectured by Klebanov and Polyakov [13] to relate the simplest AdS_4 HS theory to $3d$ $O(N)$ sigma-model and was partially proved by Giombi and Yin [14, 15] for correlators involving any three spins s_1, s_2, s_3 that do not respect the triangle inequality.

Recently, Maldacena and Zhiboedov conjectured [20] that AdS_4 HS theory is dual to the $3d$ free model even beyond the large N limit. The arguments of [20] are very general, generalizing the Coleman-Mandula theorem [99] to conformal theories. Namely, the authors of [20] have shown that if a unitary local conformal field theory possesses a conserved HS current then it must be a theory of currents of free conformal fields. Since the AdS_4 HS theory possesses HS symmetries, the conclusion of [20] was that its boundary dual is free.

Analysis of this paper shows however that, except for two particular cases, the boundary

theory dual to AdS_4 HS theory turns out to be nonlinear, escaping some of conditions of the Maldacena-Zhiboedov theorem. Namely, the boundary theory describes interactions of conformal currents in the framework of $3d$ conformal HS gauge theory which extends $3d$ (Chern-Simons) conformal gravity to higher spins. Being a gauge theory, it is not unitary, while a particular gauge choice makes it nonlocal and/or not conformal. Another property of the boundary dual of the AdS_4 HS theory is that boundary conformal currents associated with massless fields in AdS_4 are not conserved in the usual sense being instead covariantly conserved with respect to the $3d$ conformal HS algebra. Analogous phenomena are expected to take place for higher dimensions $d > 4$, relating nonlinear HS theories in any d [100] to boundary conformal HS theories in $d - 1$. However, this duality is expected to be far more complicated because of complicated structure of the corresponding generalized twistor space.

We have identified two particular truncations of the bulk HS theories which have free bosonic and fermionic boundary duals in agreement with the conjectures of Klebanov and Polyakov [13] and Sezgin and Sundell [22]. In this cases $3d$ conformal HS gauge fields decouple from the boundary currents and the corresponding boundary theories indeed turn out to be free in agreement with the Maldacena-Zhiboedov theorem [20]. Truncations to the free boundary theories are based on the parity automorphism P of the AdS_4 system that reflects the Poincaré coordinate z . Its application requires the doubling of the Poincaré chart, identifying the AdS_4 boundary with the stationary surface of P .

In the setup with doubled bulk space, it is not necessary to impose definite boundary conditions at $z = 0$ since it becomes a regular point in terms of appropriately rescaled twistor variables. Hence, in our approach, the $3d$ dual of AdS_4 HS gauge theory describes a doubled number of $3d$ currents which in particular contain two scalar currents of different dimensions. Generally, all these currents interact via $3d$ conformal HS gauge fields. We believe that the trick with doubled bulk in AdS/CFT , which follows naturally from unfolded dynamics, should also have interesting applications beyond HS theories.

A new phenomenon found in this paper is that both of holographically dual theories are theories of (conformal) gravity. This phenomenon seems to be very general and should take place beyond the $N \rightarrow \infty$ limit for most of holographic models of bulk gravity as a consequence of coordinate independence of the unfolded formulation.

In this paper we did not check explicitly how our prescription reproduces conformal correlators on the conformal side, leaving discussion of this issue to the future work [101].

Taking into account that non-Abelian contributions to conformal HS curvatures exist only for spins s_1, s_2, s_3 that respect the triangle inequality, it would be interesting to see whether nonlinear corrections of the boundary theory can help to conform the boundary and bulk calculations of [14] for such spins.

Analysis of this paper is mostly on-shell, operating in terms of field equations rather than action since HS actions are not yet known to all orders on the both sides of the HS duality. Once they are available, the analysis can be immediately extended to the action level. Hence, most urgent problems for the future include explicit construction of nonlinear $3d$ HS conformal gravity and action functionals for both AdS_4 HS theory and its $3d$ dual.

A peculiar feature emerged from the analysis of the particular HS model in this paper is

that the infinite boundary limit is not a necessary ingredient of the duality which can formally be established on every co-dimension one surface Σ in the bulk. However, for general Σ , the relation between fields and sources in the dual theories, that respects conformal symmetry, is nonlocal while in the infinite boundary limit $z \rightarrow 0$ the relation turns out to be local in accordance with the standard prescription of [5, 6]. Being complicated in terms of space-time coordinates, the nonlocal holographic duality map between two theories on general Σ acquires natural meaning in terms of non-commutative twistor variables, describing the map between Weyl and Wick star products. It should be noted however that transition from one ordering prescription to another may, in principle, lead to divergencies in the star-product formalism in HS theories because the construction involves nonpolynomial elements like Klein operators (6.19). When this happens, a model exhibits conformal anomaly.

Systematic reformulation of unfolded theories in terms of twistor variables greatly simplifies analysis of holographic duality making it nearly tautological. Seemingly different theories are described by solutions of the same equations in the generalized twistor space or by the same action-like invariants evaluated as integrals over twistor variables. Two holographically dual models result from different space-time extensions of the same twistor model.

In [9] it was conjectured that massless conformal HS theories may form a chain of dualities between models in space-times of different dimensions. If a boundary theory contains conformal gravity it can be again put in locally AdS_d background, say, by using the foliation prescription of Section 8.1. In the end, one stops at some $2d$ conformal theory, $1d$ quantum-mechanical theory or even $0d$ matrix-like theory, which is nothing but the part of the theory reduced solely to the twistor space (e.g., equations (6.7) and (6.8) in the AdS_4 HS system).

Duality between HS theories and nonrelativistic quantum mechanics discussed in Section 12 provides an exciting example of $1d$ dual interpretation. Deep relation between HS theories and quantum mechanics makes it difficult to refrain from speculation that the two systems may be literally equivalent while their different interpretations depend on particular details of physical observation in question. In other words, it is tempting to rise a risky question whether HS theories can tell us what quantum mechanics is. This issue has too many aspects to be discussed in detail in this paper. However, one immediate consequence is that, if true, nonlinear HS theories should imply that Schrodinger equation has to receive nonlinear corrections of the form prescribed by HS theory. Since the coupling constant inherited from HS theories should be related to the gravitational constant, nonlinear corrections to quantum mechanics should be negligible in the non-relativistic regime. Nevertheless one can speculate that their appearance may shed some light on such conceptual problems of quantum mechanics as, for instance, momentary wave packet reduction.

Tremendous robustness of the quantum gravity problem suggests that its solution may require modification of the both ingredients. HS theory may provide a framework for non-trivial merge of gravity with quantum mechanics, affecting the present-day understanding of both. If so, non-relativistic quantum mechanics may one day provide us with an unexpected tool for the study of quantum gravity in laboratory experiments. At any rate, we believe that HS gauge theory has potential to unify gravity and quantum mechanics in a nontrivial and constructive way.

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